

## ANOTHER LOOK AT NONPARAMETRIC ESTIMATION FOR TREND RENEWAL PROCESSES

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*Abstract* A trend renewal process is characterized by a counting process and a renewal process which are mutually transformed with each other by a trend function, and plays a significant role to represent a sub-class of general repair models. In this paper we develop another nonparametric estimation method for trend renewal processes, where the form of failure rate function in the renewal process is unknown. It is regarded as a dual approach for the nonparametric monotone maximum likelihood estimator by Heggland and Lindqvist (2007) and complements their result under the assumption that the form of trend (intensity) function is unknown. We validate our nonparametric estimator through simulation experiments and apply to a field data analysis of a repairable system.

**Keywords:** Reliability, repairable system, trend renewal process, trend function, failure rate function, nonparametric maximum likelihood estimator

### 1. Introduction

Stochastic point processes are commonly used to describe the time-to-failure phenomena in life data analysis and can be classified by the kind of failure time (lifetime) distribution and repair operation [3]. In non-repairable systems, a system or component after it fails is replaced by a new one. If the replacement time is negligible in terms of the failure time scale, it denotes a recurrent phenomenon and can be described by a *renewal process* (RP) with independent and identically distributed (i.i.d.) inter-failure time distributions [16]. On the other hand, the simplest example of repairable systems is the *minimal repair process* [3], where at each failure a repair action is performed to return the failed component state to the normal condition. In some cases, such an activity restores only damaged part of the failure component back to a working condition that is only as good as it was just before the failure. Since it is known that the minimal repair process with negligible repair time sequence is identical to a *nonhomogeneous Poisson process* (NHPP), the analytical treatment is rather easier than that in the RP. Boswell [10] proposes a nonparametric maximum likelihood estimator for NHPPs whose mean value functions are unknown. Latter, Bartozynski *et al.* [7] derive three nonparametric maximum likelihood estimation algorithms (constrained maximum likelihood estimation, penalized maximum likelihood estimation, and an application to a Cox process [16]) and apply them to the melanoma metastatic data. Barlow and Davis [6] and Kvaloy and Lindqvist [26] consider another nonparametric estimator based on the total time on test concept with multiple lifetime data sets of a repairable system. In this way, nonparametric estimation of stochastic point processes is useful to estimate the repairable systems under uncertainty when the parametric form cannot be characterized in advance.

Because the minimal repair is an ideally simplest repair operation, of course, it is difficult

to describe the complex repair activities in real world by using only this. Since the seminal contribution by Brown and Proschan [12], a number of more realistic repair models have been studied in the literature. The *imperfect repair* model by Brown and Proschan [12] branches conditionally to a replacement and a minimal repair with constant probability, and the resulting non-conditional cumulative number of failures forms an NHPP. Kijima [25] extends drastically the concept of imperfect repair and proposes a *general repair model*, which is often referred to as *the Kijima model* to celebrate his contribution. The general repair model is viewed as an intermediate repair model between replacement and minimal repair. So, this model can represent a general pattern that a repair action is performed to return an arbitrary state between new component (by replacement) and the state as good as it was just before the failure (by minimal repair). Dorado *et al.* [17] propose a nonparametric estimator for one of Kijima models and show its asymptotic properties including the uniform consistency. In fact, there exist many sub-models in the general repair model. The main concern from the statistical point of view is that they should be statistically *estimable* models with the underlying failure-repair data in real applications. Berman [8] proposes a *nonhomogeneous gamma process* (NHGP) as a gamma renewal process modulated by an NHPP\*. For application examples of NHGP in hydrology and software engineering, see [22] and [23, 42], respectively.

Although NHGP is not analytically tractable, the corresponding likelihood function is reduced to a simple form. In other words, once the failure-repair time data are given, the model parameters can be estimated easily by means of the maximum likelihood estimation, so one does not need to invent a complex frequentist approach such as Dorado *et al.* [17]. Lakey and Rigdon [27] and Black and Rigdon [9] consider a special case of the Berman's NHGP [8], where the underlying NHPP is given by a power law process [15, 40]. Though these authors call this stochastic process *the modulated power law process* (MPLP), it can be regarded as a complete parametric version based on the power law-type NHPP and a gamma renewal process. Muralidharan [33, 34] and Muralidharan *et al.* [35] examine the statistical test and reliability prediction for MPLP. Calabria and Pulcini [13, 14] apply the Bayes approach to the parameter inference and testing for MPLP. Trend renewal process (TRP) is also characterized by both of a general NHPP and a general renewal process, and is developed by Lindqvist *et al.* [29]. It involves NHGP and MPLP as special cases and plays a significant role to represent a sub-class of general repair models. Furthermore, the TRP has a variation called *heterogeneous trend renewal process* (HTRP) by taking account of *unobserved heterogeneity* which is equivalent to a random effect. Proschan [38], Bain and Wright [5], and Engelhardt and Bain [18], Lawless [28], Ng and Cook [36] consider *heterogeneous homogeneous Poisson processes* (HHPPs) and *heterogeneous nonhomogeneous Poisson processes* (HNHPPs). Follmann and Goldberg [19], Aalen and Husebye [1] discuss *heterogeneous renewal processes* (HRPs). Lindqvist *et al.* [29] show that these models are the special cases of HTRP. Recently, the TRP stopped at a random time independent from the process is considered by Badia [4]. For a good survey on TRPs, see Lindqvist [30].

Apart from the generalization of TRPs, the statistical inference is a critical issue to apply TRPs to real lifetime analysis. Jokiel-Rokita and Magiera [24] summarize the maximum

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\*The term of *gamma processes* has two different meanings. In reliability context, the gamma wear process or the gamma degradation process is often used [37, 44], where the state distribution at an arbitrary time is given by a gamma distribution with time-dependent parameter. On the other hand, the definition by Berman [8] is quite different from the above. He assumes a situation where there are  $\kappa$  ( $> 0$ ) NHPPs with exactly same intensity functions and where the observed failure epoch corresponds to every successive  $\kappa$ th event of the underlying NHPP.

likelihood estimation for parametric TRPs and introduce alternative methods of estimating model parameters in specific TRPs, which include the least squares method, constrained least squares method and moment match. Heggland and Lindqvist [21] assume a Weibull failure rate function and propose a *constrained nonparametric maximum likelihood estimator* (CNPMLE) when the time transformation function called the *trend function*, which is similar to the intensity function of an NHPP, is monotone but unknown. The basic idea is due to the CNPMLE for an NHPP by Boswell [10] and Bartozynski *et al.* [7]. In this paper we develop another CNPMLE for TRPs, under the assumption that the failure rate function of the underlying RP is unknown but the form of trend function is known. It is regarded as a dual approach for the CNPMLE by Heggland and Lindqvist [21] and complements their result under the assumption that the trend function is unknown. We apply a nonparametric maximum likelihood estimator of the failure rate function by Grenander [20] to represent the underlying RP, where it is strongly consistent (see Marshall and Proschan [31]) and possesses several rich properties on asymptotics (see Rao [39]). We provide a computation algorithm to derive CNPMLE of the TRP for a given trend function. Finally, we estimate the TRPs with two CNPMLEs and validate our nonparametric estimator through simulation experiments. We also apply our method to a field data analysis of a repairable system.

## 2. Trend Renewal Processes

Suppose that a repairable system starts operating at time  $t = 0$  and that  $n$  failure times  $T_i$  ( $T_1 < T_2 < \dots < T_n$ ), which are the random variables, are observed. Without any loss of generality, we assume  $n \geq 2$ , *i.e.*, more than one failure occur during the system operation. Once a failure occurs, a repair action is performed just after each failure to return the failed state to the normal condition. Let  $\{N(t), t \geq 0\}$  be the total number of failures occurred by time  $t$ , and let  $X_i$  ( $i = 1, 2, \dots, n$ ) be the time between  $(i - 1)$ st and  $i$ th failures, that is,  $X_i = T_i - T_{i-1}$ , where  $T_0 = 0$  and  $T_1 = X_1$  for simplicity. We also define  $t_i$  and  $x_i$  as realizations of  $T_i$  and  $X_i$ , respectively. It is assumed that each length of repair time can be negligible, so we observe only the failure epoch in continuous time. To complete the discussion, let us introduce homogeneous Poisson process (HPP), nonhomogeneous Poisson process (NHPP) and renewal process (RP) in the following:

1. *Homogeneous Poisson process*: HPP( $\lambda$ )

The stochastic point process  $\{N(t), t \geq 0\}$  is called an HPP( $\lambda$ ) if  $X_1, X_2, \dots$  are i.i.d. random variables and follow an exponential distribution with parameter  $\lambda$  ( $> 0$ ).

2. *Renewal process*: RP( $F(x)$ )

The stochastic point process  $\{N(t), t \geq 0\}$  is called an RP( $F(x)$ ) if  $X_1, X_2, \dots$  are i.i.d. and follow an arbitrary cumulative distribution function (c.d.f.)  $F(x)$ . If  $F(x)$  is an exponential distribution with parameter  $\lambda$ , then RP( $F(x)$ ) reduces to HPP( $\lambda$ ).

3. *Nonhomogeneous Poisson process*: NHPP( $\lambda(t)$ )

Let  $\lambda(t)$  ( $t \geq 0$ ) be a nonnegative and absolutely continuous function, and represent a deterministic intensity function of the process. Then the cumulative intensity (mean value) function is defined by  $\Lambda(t) = \int_0^t \lambda(s) ds$ . Suppose that there exists an inverse function of  $\Lambda(t)$ , say  $\Lambda(t)^{-1}$ . The stochastic point process  $\{N(t), t \geq 0\}$  is called an NHPP( $\lambda(t)$ ) if the time-transformed process  $\Lambda(T_1), \Lambda(T_2), \dots$  is an HPP(1).

Next we define the trend-renewal process (TRP) as a generalization of HPP( $\lambda$ ), RP( $F(x)$ ) and NHPP( $\lambda(t)$ ).

4. *Trend renewal process:* TRP( $F(x), \lambda(t)$ )

Let  $\lambda(t)$  and  $\Lambda(t)$  be the intensity and cumulative intensity functions of the process, respectively. The stochastic point process  $\{N(t), t \geq 0\}$  is called a TRP( $F(x), \lambda(t)$ ) if the time-transformed process,  $\Lambda(T_1), \Lambda(T_2), \dots$ , is an RP( $F(x)$ ). Then the c.d.f.  $F(x)$  is called the renewal distribution, and  $\lambda(t)$  is called the trend function of the TRP.

From the above definition, it can be easily checked that the TRP generalizes both NHPP and RP, since TRP( $1 - e^{-x}, \lambda(t)$ ) is equivalent to NHPP( $\lambda(t)$ ), and TRP( $F(x), 1$ ) is equivalent to RP( $F(x)$ ). It is also worth mentioning that the representation of TRP( $F(x), \lambda(t)$ ) is not always unique, since TRP( $F(x), \lambda(t)$ ) equals TRP( $F(cx), \lambda(t)/c$ ) for an arbitrary constant  $c > 0$  (see Lindqvist *et al.* [29]). This fact implies that a scale factor such as the parameter  $c$  should be removed from  $F(x)$  (or the corresponding failure rate function) and  $\lambda(t)$ , to get a canonical representation.

More formally, the conditional intensity function of an arbitrary point process (see Andersen *et al.* [2] and Bremaud [11]) is defined by

$$\zeta(t) = \lim_{\delta t \rightarrow 0} \frac{P(\text{failure in } [t, t + \delta t] \mid \Psi_{t-})}{\delta t}, \tag{2.1}$$

where  $\Psi_{t-}$  denotes the history of the process  $\{N(t), t \geq 0\}$  up to, but not including time  $t$ . In general, the conditional intensity function will be no longer deterministic and depends on the past history of events. For example, the conditional intensity function for RP( $F(x)$ ) is given by  $\zeta(t) = r(t - T_{N(t-)})$ , where  $r(x)$  is the failure rate function corresponding to renewal distribution  $F(x)$ , *i.e.*,  $r(x) = (dF(x)/dx)/(1 - F(x))$ . In the case of NHPP( $\lambda(t)$ ), the conditional intensity function is not probabilistic and is represented by a deterministic function  $\zeta(t) = \lambda(t)$ . The conditional intensity function of TRP( $F(x), \lambda(t)$ ) is given by Lindqvist [29]:

$$\begin{aligned} \zeta(t) &= \lim_{\delta t \rightarrow 0} \frac{P(\text{failure follows RP}(F(x)) \text{ in } [\Lambda(t), \Lambda(t + \delta t)] \mid \Psi_{t-})}{\delta \Lambda(t)} \cdot \frac{\delta \Lambda(t)}{\delta t} \\ &= r(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda(t). \end{aligned} \tag{2.2}$$

When  $n$  failure times,  $t_1, t_2, \dots, t_n$ , which are generated by  $\{N(t), t \geq 0\}$  with conditional intensity function  $\zeta(t)$ , are observed, the likelihood function of the stochastic point process is given by

$$LF = \left\{ \prod_{i=1}^n \zeta(t_i) \right\} \exp \left\{ - \int_0^{t_n} \zeta(u) du \right\} \tag{2.3}$$

(see [2, 11]). By substituting Equation (2.2) into Equation (2.3), the likelihood function of a TRP is obtained as follows:

$$LF = \left\{ \prod_{i=1}^n r(\Lambda(t_i) - \Lambda(t_{i-1}))\lambda(t_i) \right\} \exp \left\{ - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} r(\Lambda(u) - \Lambda(t_{i-1}))\lambda(u) du \right\}. \tag{2.4}$$

By making the substitution  $v = \Lambda(u) - \Lambda(t_{i-1})$  and taking the logarithm of both sides in Equation (2.4), we get the log-likelihood function of TRP:

$$LLF = \sum_{i=1}^n \left\{ \log \left( r(\Lambda(t_i) - \Lambda(t_{i-1})) \right) + \log(\lambda(t_i)) - \int_0^{\Lambda(t_i) - \Lambda(t_{i-1})} r(v) dv \right\}. \tag{2.5}$$

### 3. Nonparametric Estimation of Trend Function

#### 3.1. Constrained nonparametric maximum likelihood estimator (CNPMLE)

First we overview the CNPMLE of  $\lambda(t)$  by Heggland and Lindqvist [21]. The objective here is to derive an estimator which maximizes the log-likelihood function given in Equation (2.5) under the condition that  $\lambda(t)$  belongs to a class of nonnegative monotone functions on  $(0, t_n]$ , where *monotone* means *nondecreasing* or *nonincreasing* in time. To avoid confusions, we mean through this paper “increasing” for “nondecreasing” and “decreasing” for “nonincreasing”, respectively. Heggland and Lindqvist [21] make some assumptions on the failure rate function  $r(x)$  to derive the CNPMLE of trend function  $\lambda(t)$ . It is difficult to obtain maximum likelihood (ML) estimates of both  $r(x)$  and  $\lambda(t)$  simultaneously, except in the case where  $r(x)$  is constant and the TRP reduces to an NHPP. Heggland and Lindqvist [21] consider a case where the form of failure rate function  $r(x)$  is given but the trend function  $\lambda(t)$  is unknown. For instance, they suppose that the underlying renewal process is given by a Weibull renewal process with unknown parameter. To distinguish from MPLP [9, 27], we call this stochastic point process the *nonhomogeneous power law process* (NHPLP) in this paper. Let  $r(x)$  be the Weibull failure rate function:

$$r(x) = ab(ax)^{b-1}, \quad (a, b > 0). \quad (3.1)$$

Here, we put the scale parameter  $a = 1$  to guarantee the uniqueness of the TRP. Suppose that  $n$  failure times,  $t_1, t_2, \dots, t_n$ , which are measured from time  $t = 0$ , are available. Then, the likelihood function of NHPLP is given by

$$LF_1^W = \prod_{i=1}^n \lambda(t_i) b \left\{ \Lambda(t_i) - \Lambda(t_{i-1}) \right\}^{b-1} \exp \left( - \sum_{i=1}^n \left\{ \Lambda(t_i) - \Lambda(t_{i-1}) \right\}^b \right). \quad (3.2)$$

By taking the logarithm of both sides in Equation (3.2), Heggland and Lindqvist [21] obtain the log-likelihood function for NHPLP:

$$LLF_1^W = \sum_{i=1}^n \left[ \log \lambda(t_i) + (b-1) \log \left\{ \Lambda(t_i) - \Lambda(t_{i-1}) \right\} - \left\{ \Lambda(t_i) - \Lambda(t_{i-1}) \right\}^b \right] + n \log b, \quad (3.3)$$

where  $t_0 = \Lambda(t_0) = 0$ .

Saito and Dohi [42] consider a somewhat different TRP from Heggland and Lindqvist [21]. Similar to Berman's NHGP [8], they assume that the underlying RP is a gamma RP with the probability density function (p.d.f.)  $g(x)$  with unit scale parameter:

$$g(x) = \frac{x^{\kappa-1} \exp(-x)}{\Gamma(\kappa)}, \quad (3.4)$$

where  $\Gamma(\cdot)$  denotes the standard gamma function. In the case of Berman's NHGP, it is evident that  $\Lambda(t_{i+1}) - \Lambda(t_i)$  obeys a renewal process having the gamma renewal distribution  $G(x) = \int_0^x g(s) ds$  with integer-valued shape parameter  $\kappa$  and unit scale parameter. Then, the likelihood function of NHGP is given by a

$$LF_1^G = \prod_{i=1}^n \frac{\lambda(t_i) \left\{ \Lambda(t_i) - \Lambda(t_{i-1}) \right\}^{\kappa-1}}{\Gamma(\kappa)} \exp(-\Lambda(t_n)). \quad (3.5)$$

Taking the logarithm of both sides in Equation (3.5) yields the log-likelihood function for NHGP:

$$LLF_1^G = \sum_{i=1}^n \left[ \log \lambda(t_i) + (\kappa - 1) \log \left\{ \Lambda(t_i) - \Lambda(t_{i-1}) \right\} \right] - \Lambda(t_n) - n \log \Gamma(\kappa). \quad (3.6)$$

It is worth noting that the role of integer-valued shape parameter  $\kappa$  gives a physical meaning of NHGP as the number of successive events to represent a failure but is not essential, though Berman [8] restricts only the integer-valued shape parameter. For example, when  $\kappa > 1$ , the system is improved in better condition just before the failure, and the larger  $\kappa$  indicates the larger improvement effect. On the other hand, if  $\kappa < 1$ , then the system gets worse just after the occurrence of system failures. In a fashion similar to NHGP, the shape parameter  $b$  for NHPLP plays the similar role to  $\kappa$ . When  $\kappa$  (or  $b$ )  $\neq 1$ , the conditional intensity function in Equation (2.2) is stochastic and depends on the past history of events. Also, it turns out that  $\Lambda(t)$  does not always equal to the mean value of these processes,  $E[N(t)]$  when  $\kappa$  (or  $b$ )  $\neq 1$ . Saito and Dohi [42] develop a similar CNPMLE of an NHGP with integer-valued shape parameter  $\kappa$  to Heggland and Lindqvist [21]. However, we relax this assumption hereafter and deal with an NHGP having a real-valued shape parameter  $\kappa$ , in addition to NHPLP. When  $\kappa$  (or  $b$ ) = 1, it is seen that both of NHGP and NHPLP can be reduced to an NHPP. This implies that the system would not be as same as it was just before the failure when  $\kappa$  (or  $b$ )  $\neq 1$ . Furthermore, if the trend function is a constant ( $\lambda(t) = \rho$ ), the NHGP and NHPLP become RPs with the gamma and Weibull renewal distributions with shape parameter  $\kappa$  ( $b$ ) and scale parameter  $\rho$ . In this case, the system after repairs would be as good as new one. Hence two parametric models, NHGP and NHPLP, are also characterized as non-stationary stochastic point processes with the trend function  $\lambda(t)$ . In the following discussion, we implicitly assume NHGP or NHPLP as a parametric form.

### 3.2. Case of increasing $\lambda(t)$

In what follows, we concern the TRP, whose trend function in the NHPP component,  $\lambda(t)$ , is monotone, and derive the CNPMLE. Suppose that the NHPLP or NHGP has an increasing trend function for a given (Weibull or gamma) failure rate in the RP component. Note that Heggland and Lindqvist [21] just consider only the NHPLP. The problem here is to maximize the log-likelihood function in Equation (3.3) or (3.6) with respect to  $\lambda(t)$ . Then it can be seen in [21] that the maximizer of the log-likelihood function for any failure rate function must consist of step functions closed on the left with no jumps except at some of the failure time points. Let  $\lambda_i = \lambda(t_i)$  ( $i = 0, 1, \dots, n$ ) and  $x_i = t_i - t_{i-1}$  ( $i = 1, 2, \dots, n$ ). Since  $\Lambda(t_i) - \Lambda(t_{i-1})$  can be represented by  $\lambda_{i-1}x_i$ , the likelihood function for NHPLP in Equation (3.2) can be rewritten by

$$LF^W(b, \lambda_0, \lambda_1, \dots, \lambda_n) = \left[ \prod_{i=1}^n \lambda_i b (\lambda_{i-1} x_i)^{b-1} \right] \exp \left( - \sum_{i=1}^n (\lambda_{i-1} x_i)^b \right). \quad (3.7)$$

The maximization problem of Equation (3.7) subject to  $0 \leq \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_n$  is equivalent to maximize

$$\begin{aligned} LLF^W(b, \lambda_0, \dots, \lambda_n) &= n \log b + (b - 1) \sum_{i=1}^n \log x_i + (b - 1) \log \lambda_0 - \lambda_0^b x_1^b \\ &\quad + \sum_{i=1}^{n-1} \{ b \log \lambda_i - \lambda_i^b x_{i+1}^b \} + \log \lambda_n. \end{aligned} \quad (3.8)$$

Similar to NHPLP, the problem of maximizing Equation (3.5) can be reduced to maximizing

$$LF^G(\kappa, \lambda_0, \lambda_1, \dots, \lambda_n) = \left[ \prod_{i=1}^n \lambda_i \lambda_{i-1}^{\kappa-1} x_i^{\kappa-1} \right] \frac{\exp(-\sum_{i=1}^n \lambda_{i-1} x_i)}{\Gamma(\kappa)^n}, \tag{3.9}$$

subject to  $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_n$ . This is also equivalent to maximize

$$\begin{aligned} LLF^G(\kappa, \lambda_0, \lambda_1, \dots, \lambda_n) &= -n \log \Gamma(\kappa) + (\kappa - 1) \sum_{i=1}^n \log x_i + (\kappa - 1) \log \lambda_0 - \lambda_0 x_1 \\ &+ \sum_{i=1}^{n-1} \{ \kappa \log \lambda_i - \lambda_i x_{i+1} \} + \log \lambda_n. \end{aligned} \tag{3.10}$$

For the maximization problems in Equations (3.8) and (3.10), it is evident that the log-likelihood functions increase endlessly in  $\lambda_n$ . Hence, to guarantee a finite likelihood value, we suppose that there exists an upper bound of  $\lambda_n$ , *i.e.*,  $\lambda_n \leq m < \infty$ . Suppose at the moment that the values of shape parameter  $b$  in NHPLP and  $\kappa$  in NHGP are given. For the maximization problem in Equation (3.8) (Equation (3.10)) under a given  $b$  ( $\kappa$ ), we obtain the maximum likelihood (ML) estimates of  $\lambda_i$  by solving the following maximization problem:

$$\max_{w_0, w_1, \dots, w_n} \sum_{i=0}^n \{ z_i \log w_i - y_i w_i \}, \tag{3.11}$$

subject to  $0 \leq w_0 \leq w_1 \leq \dots \leq w_n$ , where  $y_i = x_{i+1}^b$  ( $i = 0, 1, \dots, n - 1$ ),  $y_n = 0$ ,  $z_0 = b - 1$ ,  $z_i = b$  ( $i = 1, 2, \dots, n - 1$ ),  $z_n = 1$  and  $w_i = \lambda_i^b$  ( $i = 0, 1, \dots, n$ ) for NHPLP ( $y_i = x_{i+1}$  ( $i = 0, 1, \dots, n - 1$ ),  $y_n = 0$ ,  $z_0 = \kappa - 1$ ,  $z_i = \kappa$  ( $i = 1, 2, \dots, n - 1$ ),  $z_n = 1$  and  $w_i = \lambda_i$  ( $i = 0, 1, \dots, n$ ) for NHGP). All  $y_i$  and  $z_i$  are always positive except that  $z_0$  may take zero or negative. For a large enough  $m$ , the solution to this problem is given by

$$w_0 = w_1 = \dots = w_{j_1} = \min_{0 \leq t \leq n-1} \frac{\sum_{i=0}^t z_i}{\sum_{i=0}^t y_i}, \tag{3.12}$$

$$w_{j_1+1} = w_{j_1+2} = \dots = w_{j_2} = \min_{j_1+1 \leq t \leq n-1} \frac{\sum_{i=j_1+1}^t z_i}{\sum_{i=j_1+1}^t y_i}, \tag{3.13}$$

⋮

where  $j_1$  is the time  $t$  minimizing the last term of Equation (3.12) and  $j_2 (> j_1)$  is the time  $t$  minimizing the last term of Equation (3.13), *etc.* We repeat this procedure until  $j_l = n - 1$ . In the latter discussion we set  $w_n = m^b$  for NHPLP and  $w_n = m$  for NHGP.

Heggland and Lindqvist [21] mention some remarks on the above solution in NHPLP. If  $b$  (or  $\kappa$ )  $< 1$ ,  $w_0$  will attain a negative value because  $z_0$  becomes less than zero. Since  $w_0 \geq 0$  is assumed implicitly, the optimum value of  $w_0$  will be 0 in this case. However, unfortunately, it gives  $\max LLF^W$  ( $LLF^G$ )  $\rightarrow -\infty$ . This contradicts the ML principle. Heggland and Lindqvist [21] introduce an artificial constraint  $w_0 \geq \delta > 0$  upon the problem. This will give  $w_0 = \delta$  as a solution for keeping a finite likelihood, but will not affect any of the other  $w_i$  if we choose  $\delta$  small enough. As mentioned above, the log-likelihood can be increased as  $m$  approaches to infinity. If a finite  $m$  is given, we use it directly, otherwise, the finite ML estimate on  $\lambda_i$  does not exist. In such a case, it may work better to assume  $m = \lambda_{n-1}$

to get a pessimistic estimation result for goodness-of-fit performance in the sense of ML estimation.

Next, we consider the maximization problem of Equation (3.8) (Equation (3.10)) when  $b(\kappa)$  is unknown. First set  $b = b^{(1)}$  ( $\kappa = \kappa^{(1)}$ ), where  $b^{(1)}$  ( $\kappa^{(1)}$ ) is an initial guess. We compute  $y_i$  and  $z_i$ , and find the solutions,  $w_0^{(1)}, w_1^{(1)}, \dots, w_n^{(1)}$ , from Equations (3.12) and (3.13). Then we obtain an estimate of  $\lambda(t)$ ;  $\lambda_i^{(1)} = (w_i^{(1)})^{1/b^{(1)}}$  for NHPLP and  $\lambda_i^{(1)} = w_i^{(1)}$  for NHGP. Second, we update  $b = b^{(1)}$  ( $\kappa = \kappa^{(1)}$ ) to  $b^{(2)}$  ( $\kappa^{(2)}$ ). By the definition of TRP, since  $R_i = \Lambda(t_i)$  ( $i = 1, 2, \dots, n$ ) follows an RP( $F(x)$ ), we can define:

$$R_i^{(1)} = \sum_{l=0}^{i-1} \lambda_l^{(1)} x_{l+1}, \quad (i = 1, 2, \dots, n). \tag{3.14}$$

Under the assumption for NHPLP which is an RP( $F(x)$ ) having the Weibull renewal distribution,  $F(x) = 1 - \exp(-x^b)$ , the profile likelihood function is given by

$$LF^W = \prod_{i=1}^n f(R_i^{(1)} - R_{i-1}^{(1)}), \tag{3.15}$$

where  $f(x)$  is the p.d.f. of renewal distribution  $F(x)$ . Then the profile log-likelihood function for unknown  $b$  can be represented by

$$LLF^W(b) = n \log b + (b - 1) \sum_{i=1}^n \log[\lambda_{i-1}^{(1)} x_i] - \sum_{i=1}^n [\lambda_{i-1}^{(1)} x_i]^b, \tag{3.16}$$

which corresponds to Equation (3.8) except for a constant term with  $\lambda_0^{(1)}, \lambda_1^{(1)}, \dots, \lambda_{n-1}^{(1)}$ .

In the case of NHGP, since the stochastic process  $\Lambda(T_i) - \Lambda(T_{i-1})$  ( $i = 1, 2, \dots, n$ ) is a gamma RP with unit scale parameter in Equation (3.4), the profile likelihood function becomes

$$LF^G = \prod_{i=1}^n g(R_i^{(1)} - R_{i-1}^{(1)}). \tag{3.17}$$

Then the profile log-likelihood function for unknown  $\kappa$  can be written by

$$LLF^G(\kappa) = (\kappa - 1) \sum_{i=1}^n \log \lambda_{i-1}^{(1)} x_i - \sum_{i=1}^n \lambda_{i-1}^{(1)} x_i - n \log \Gamma(\kappa), \tag{3.18}$$

which also corresponds to Equation (3.10) except the constant term.

By maximizing Equation (3.16) (Equation (3.18)) with respect to  $b(\kappa)$ , an updated estimate  $b^{(2)}$  ( $\kappa^{(2)}$ ) can be obtained numerically. Concretely speaking, use  $b^{(2)}$  ( $\kappa^{(2)}$ ) to obtain an update  $\lambda_i^{(2)}$  ( $i = 0, 1, \dots, n$ ) for the trend function  $\lambda(t)$ , and find an update  $b^{(3)}$  ( $\kappa^{(3)}$ ) for  $b(\kappa)$  with  $\lambda_i^{(2)}$ . This procedure repeats until the difference between two successive estimates of  $b(\kappa)$  is smaller than a given tolerance level  $\epsilon$ . In this procedure, if  $\lambda_0^{(l)} = 0$ , then the TRP may be viewed as if it starts at time  $t = t_1$  but not time  $t = 0$ , so the summation operation begins from  $i = 2$  to avoid any confusion.

Heggland and Lindqvist [21] give some remarks on the ML estimate for NHPLP. To overcome a technical problem on convergence in the iteration scheme, the following heuristic rule is proposed by Heggland and Lindqvist [21]: Set  $\lambda_0 = 0$  in the initial phase and ignore

the term  $(b - 1) \log \lambda_0$  at the moment. If the value of  $b$  converges to a value less than 1, then continue using the current estimates. If not, return back again to the initial step after including the term  $(b - 1) \log \lambda_0$ . If the iteration converges to a value greater than 1, then resulting estimate is applied, otherwise, the previous estimate can be used. This idea can be also used in estimating  $\kappa$  for NHGP. In our experiences, when the term  $(b - 1) \log \lambda_0$  is included in calculation and the shape parameter converges to a value less than 1, the resulting estimates based on Heggland and Lindqvist's rule are not optimal in many cases. In other words, their heuristic rule does not work well to find the ML estimates for the general case. Here we propose an alternative idea to stabilize the algorithm. Similar to the assumption of monotonicity for trend function, it can be validated in many applications to assume the monotonicity for failure rate function for the underlying renewal distribution. For instance, if the renewal distribution is increasing failure rate (IFR) in NHPLP, then the shape parameter is always estimated as  $b \geq 1$ . Conversely, if the renewal distribution is decreasing failure rate (DFR), the solution is given as  $b \leq 1$ . In the remaining part of this paper, we pay our attention for the monotone trend function and monotone failure rate function.

### 3.3. Case of decreasing $\lambda(t)$

The CNPMLE with decreasing trend function is used for estimating software reliability [23, 42] to represent the so-called *reliability growth phenomena*. Though the method of obtaining a CNPMLE with decreasing  $\lambda(t)$  is very similar to that with increasing  $\lambda(t)$ , we quickly summarize it to complete our discussion. Even in this case, the resulting ML estimate also consists of step functions with jumps only at any subset of the failure times which is not left continuous but right continuous. Let  $\lambda_i = \lambda(t_i)$  and  $x_i = t_i - t_{i-1}$  ( $i = 1, 2, \dots, n$ ). Then, the log-likelihood function of TRP in Equation (2.5) can be simplified to the following function:

$$LLF = \sum_{i=1}^n \left\{ \log r(\lambda_i x_i) + \log \lambda_i - \int_0^{\lambda_i x_i} r(v) dv \right\}, \quad (3.19)$$

where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ . Then, the log-likelihood functions for NHPLP and NHGP are represented by

$$LLF^W(b, \lambda_1, \dots, \lambda_n) = n \log b + (b - 1) \sum_{i=1}^n \log x_i + \sum_{i=1}^n \{ \log \lambda_i^b - \lambda_i^b x_i^b \}, \quad (3.20)$$

$$LLF^G(\kappa, \lambda_1, \dots, \lambda_n) = -n \log \Gamma(\kappa) + (\kappa - 1) \sum_{i=1}^n \log x_i + \sum_{i=1}^n \{ \log \lambda_i^\kappa + \lambda_i x_i \}, \quad (3.21)$$

respectively. We apply the similar iteration scheme to Subsection 3.2 to obtain ML estimates of  $b$  ( $\kappa$ ) and  $\lambda_i$ , where the only difference from the increasing trend function is to solve the following maximization problem:

$$\max_{w_1, w_2, \dots, w_n} \sum_{i=1}^n \{ z_i \log w_i - w_i y_i \}, \quad (3.22)$$

subject to  $w_1 \geq w_2 \geq \dots \geq w_n \geq 0$ , where  $y_i = x_i^b$ ,  $w_i = \lambda_i^b$  and  $z_i = b$  for NHPLP, while  $y_i = x_i$ ,  $w_i = \lambda_i$  and  $z_i = \kappa$  for NHGP. The optimal solution can be obtained by replacing

the min operator in the increasing case with the max operator:

$$w_1 = w_2 = \dots = w_{j_1} = \max_{1 \leq t \leq n} \frac{\sum_{j=1}^t z_j}{\sum_{j=1}^t y_j}, \tag{3.23}$$

$$w_{j_1+1} = w_{j_1+2} = \dots = w_{j_2} = \max_{j_1+1 \leq t \leq n} \frac{\sum_{j=j_1+1}^t z_j}{\sum_{j=j_1+1}^t y_j}, \tag{3.24}$$

⋮

where  $j_i$  ( $i = 1, 2, \dots, l$ ) are determined by the similar manner to the case of increasing  $\lambda(t)$ . It should be noted that the decreasing case is more tractable than the increasing case, because the problem on an infinite likelihood value does not occur regardless of the choice of shape parameter  $b$  ( $\kappa$ ) or  $\lambda_i$ .

#### 4. Another Nonparametric Estimation for Trend Renewal Processes

##### 4.1. Constrained nonparametric maximum likelihood estimator (CNPMLE)

We develop another CNPMLE for the TRP, where the form of trend function is known but the failure rate function is unknown. From Equation (2.4), the problem is to derive the CNPMLE of the failure rate  $r(x)$  for a given  $\Lambda(t)$ . In the simplest i.i.d. case, *i.e.*, the TRP is reduced to an RP, Grenander [20] gives an ML estimator of the failure rate under the assumption that there is a sample of  $n$  independent observations from a c.d.f. Marshall and Proschan [31] prove the strong consistency in both IFR and DFR cases. Rao [39] derives the asymptotic distribution of the above ML estimator. Let  $r(x) = f(x)/[1 - F(x)]$  be the failure rate function of a c.d.f.  $F(x)$ , where  $F(x) < 1$  for all finite  $x$  and  $f(x)$  is the p.d.f. Suppose that  $r(x)$  is finite on  $(0, x)$  for any c.d.f.  $F(x)$ . Then we have the well-known identity:

$$1 - F(x) = \exp[-R(x)] = \exp\left[-\int_0^x r(s)ds\right]. \tag{4.1}$$

For the i.i.d. random variates  $X_i$  ( $i = 1, 2, \dots, n$ ) from a c.d.f.  $F(x)$ , let  $X_{(i)}$  ( $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ ) be the ordered random variables obtained from an unknown c.d.f.  $F(x)$ . We also define  $x_i$  and  $x_{(i)}$  as realizations of  $X_i$  and  $X_{(i)}$ , respectively. Then the likelihood function in this simple case is given by

$$LF = \prod_{i=1}^n f(x_i). \tag{4.2}$$

##### 4.2. Monotone estimator of $r(x)$

If  $F(x)$  is IFR, then  $R(x) = \int_0^x r(s)ds = -\log[1 - F(x)]$  is convex on  $[0, \infty)$  which is the support of  $F(x)$ . Let  $F(x)^{\mathfrak{S}}$  be an unknown c.d.f. belonging to the class of IFR distributions. Since  $f(x_{(n)})$  can be taken as arbitrarily large value for  $F(x)^{\mathfrak{S}}$ , it is impossible to obtain the MLE maximizing Equation (4.2) directly. Hence, we first consider a subclass  $F(x)^{\mathfrak{S}^m}$  of c.d.f.  $F(x)^{\mathfrak{S}}$  with failure rate bounded by  $m$  ( $> 0$ ). From Equation (4.1), the log-likelihood function for  $F(x)^{\mathfrak{S}^m}$  is given by

$$LLF = \sum_{i=1}^n \log \frac{dR(x)}{dx} \Big|_{x=x_i} + \sum_{i=1}^n (-R(x_i))$$

$$= \sum_{i=1}^n \log r(x_i) - \sum_{i=1}^n \int_0^{x_i} r(s) ds. \quad (4.3)$$

The following result is useful to derive the ML estimator for the i.i.d. samples.

*Proposition 4.1:* Let  $r(x)$  be the increasing failure rate bounded by  $m$  such that  $r(x_{(n)}) = m$ . Then the log-likelihood function in Equation (4.3) can be represented by

$$LLF(F) = \sum_{i=1}^{n-1} \log r(x_{(i)}) + \log m - \sum_{i=1}^n \int_0^{x_{(i)}} r(s) ds. \quad (4.4)$$

Then  $dLLF(F)/dm = 1/m > 0$  for  $m > 0$  and the log-likelihood function is increasing monotonically as  $m$  approaches to infinity.

The proof is omitted for simplicity. It can be seen that  $LLF(F)$  is maximized with respect to failure rate  $r(x)$ , which is always constant between observations  $x_{(i+1)}$  and  $x_{(i)}$ . More precisely, the optimal  $r(x)$  which maximizes the likelihood function must consist of step functions closed on the left with no jumps except at any subset of the time points  $x_{(i)}$ . Let  $F(x)^{\mathfrak{S}}$  have an arbitrary increasing failure rate  $r(x)$  and let  $\tilde{F}(x)^{\mathfrak{S}}$  be the c.d.f. with the following failure rate function:

$$\tilde{r}(x) = \begin{cases} 0, & x < x_{(1)}, \\ r(x_{(i)}), & x_{(i)} \leq x < x_{(i+1)}, \quad (i = 1, 2, \dots, n-1), \\ r(x_{(n)}), & x \geq x_{(n)}. \end{cases} \quad (4.5)$$

If  $r(x) \geq \tilde{r}(x)$  for all  $x$ , i.e.,  $-\int_0^{x_{(i)}} r(s) ds \leq -\int_0^{x_{(i)}} \tilde{r}(s) ds$  and  $r(x_{(i)}) = \tilde{r}(x_{(i)})$  for all  $i$ , it can be easily checked that  $LLF(F) \leq LLF(\tilde{F})$ , where  $F$  and  $\tilde{F}$  denotes  $F(x)^{\mathfrak{S}}$  and  $\tilde{F}(x)^{\mathfrak{S}}$ , respectively. Thus,  $LLF(F)$  in Equation (4.3) can be represented by

$$LLF(\tilde{F}) = \sum_{i=1}^n \log r(x_{(i)}) - \sum_{i=1}^{n-1} (n-i)r(x_{(i)})(x_{(i+1)} - x_{(i)}). \quad (4.6)$$

Grenander [20] solves the maximization problem of Equation (4.6) subject to  $r(x_{(1)}) \leq r(x_{(2)}) \leq \dots \leq r(x_{(n)}) = m$ . For a large enough  $m$ , the ML estimator for  $r(x)$  (corresponding to  $F(x)^{\mathfrak{S}^m}$ ) is then given by

$$\hat{r}^m(x_{(i)}) = \min_{v \geq i+1} \max_{u \leq i} \left\{ \frac{v-u}{J(u,v)} \right\}, \quad (4.7)$$

where  $\hat{r}^m(x_{(n)}) = m$  and

$$J(u,v) = \sum_{i=u}^{v-1} \{(n-i)(x_{(i+1)} - x_{(i)})\}. \quad (4.8)$$

The maximization procedure which yields Equation (4.7) is described as follows. First, we find the ML estimate which maximizes each term of Equation (4.6). If the order condition is violated, say,  $r(x_{(i)}) > r(x_{(i+1)})$ , then set  $r(x_{(i)}) = r(x_{(i+1)})$ . A monotone estimator is obtained by repeating this procedure. We can obtain the maximum value of Equation (4.6) with  $r(x_{(i)}) = r(x_{(i+1)})$  in replacement of them by their harmonic mean  $(r(x_{(i)})^{-1} +$

$r(x_{(i+1)})^{-1})^{-1}$ . By repeating the similar steps, we get the resulting estimator for a large enough  $m$ :

$$\hat{r}^m(x) = \begin{cases} 0, & x < x_{(1)}, \\ r_{k_i+1, k_{i+1}}, & x_{(k_i+1)} \leq x < x_{(k_{i+1}+1)}, \\ m, & x \geq x_{(n)}, \end{cases} \tag{4.9}$$

where  $r_{1, k_1} \leq r_{k_1+1, k_2} \leq \dots \leq r_{k_{l-1}+1, k_l}$  ( $0 = k_0 < k_1 < \dots < k_l$ ), and  $r_{k_i+1, k_{i+1}}$  is the harmonic mean of  $r_{k_i+1}, r_{k_i+2}, \dots, r_{k_{i+1}}$ . The  $k_i$  are determined so as to satisfy the order condition. By letting  $m \rightarrow \infty$  in Equation (4.9), we obtain the ML estimator for  $r(x)$  corresponding to  $F(x)^{\mathfrak{S}}$  with the unbounded failure rate:

$$\hat{r}(x) = \begin{cases} 0, & x < x_{(1)}, \\ \min_{v \geq i+1} \max_{u \leq i} \left\{ \frac{v-u}{J(u,v)} \right\}, & x_{(i)} \leq x < x_{(i+1)}, \quad (i = 1, 2, \dots, n-1), \\ \infty, & x \geq x_{(n)}. \end{cases} \tag{4.10}$$

When  $m$  is given, the CNPMLE of failure rate function is determined uniquely and the finiteness of likelihood function is guaranteed. On the other hand, when  $m$  is unknown, the likelihood may go to infinity. However it does not mean that the goodness-of-fit performance with the resulting CNPMLE is best in the sense of ML estimation. Similar to the previous discussion, it is common to assume that  $\hat{r}(x_{(n)})$  is estimated as  $\hat{r}(x_{(n-1)})$ .

Similar to the case of CNPMLE of trend function, we consider the case of decreasing  $r(x)$ . The c.d.f.  $F(x)$  is said to be DFR if  $\log[1 - F(x)]$  is convex on  $[0, \infty)$ . Given  $x_{(1)} < x_{(2)} < \dots < x_{(n)}$ , the log-likelihood function can be represented as Equation (4.3). The ML estimator maximizing Equation (4.3) under DFR assumption is given by

$$\hat{r}(x) = \begin{cases} \max_{v \geq i+1} \min_{u \leq i} \frac{v-u}{J(u,v)}, & x_{(i-1)} < x \leq x_{(i)}, \quad (i = 1, 2, \dots, n), \\ \delta, & x > x_{(n)}, \end{cases} \tag{4.11}$$

where  $x_{(0)} = 0$  and  $\delta$  is an arbitrary real number satisfying  $\delta < \hat{r}(x_{(n)})$ . Contrary to the IFR case, DFR estimator is not unique, since it is determined by the likelihood function for only  $x \leq x_{(n)}$ . Hence we must choose  $\delta$  carefully, so as to preserve the DFR property.

### 4.3. CNPMLE of TRPs

In the case of TRP,  $X_i$  is characterized by a cumulative trend function  $\Lambda(t)$ . That is, if we assume the form of a trend function, we can calculate  $\Lambda(T_i) - \Lambda(T_{i-1})$ . Let  $X_i = \Lambda(T_i) - \Lambda(T_{i-1})$  ( $i = 1, 2, \dots, n$ ), where  $\Lambda(T_0) = 0$ . Note that if the cumulative trend function changes, then the corresponding random samples  $X_i$  also change. This property has to be taken into consideration to obtain the CNPMLE of failure rate function  $r(x)$  for TRP. For better understanding, *e.g.*, we assume the power law type trend function as a special case:

$$\Lambda(t) = \alpha t^\beta, \quad (\alpha, \beta > 0), \tag{4.12}$$

where  $\alpha$  and  $\beta$  are parameters of the trend function  $\lambda(t) = \alpha \beta t^{\beta-1}$ . Due to the non-uniqueness of TRP, we set a redundant parameter  $\alpha = 1$ , since  $\text{TRP}(F(x), \alpha \beta t^{\beta-1})$  has the same maximum log-likelihood function as  $\text{TRP}(F(\alpha x), \beta t^{\beta-1})$ . Hence the CNPMLE of  $r(x)$  for  $\text{TRP}(F(x), \beta t^{\beta-1})$  is obtained by multiplying CNPMLE of  $r(x)$  for  $\text{TRP}(F(x), \alpha \beta t^{\beta-1})$  by  $\alpha$ . This fact means that the parameter  $\alpha$  of trend function does not influence to the

value of log-likelihood function of TRP. Our purpose here is to estimate model parameter  $\beta$  of trend function which maximizes the log-likelihood function of TRP:

$$LLF = \sum_{i=1}^n \left\{ \log(r(x_i)) + \log(\lambda(t_i)) - \int_0^{x_i} r(v) dv \right\}, \quad (4.13)$$

where  $x_i = \Lambda(t_i) - \Lambda(t_{i-1})$  ( $i = 1, 2, \dots, n$ ). As mentioned above, once the parameter  $\beta$  is fixed,  $x_{(i)}$  ( $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ ) are determined. Thus, we can get the CNPMLE of failure rate function  $r(x)$  by Equation (4.10) (or Equation (4.11)). In the case of CNPMLE of failure rate function  $r(x)$ , we cannot apply the iteration scheme dissimilar to Section 3. This is because if  $\beta$  is updated through the iteration scheme then not only the random samples  $X_i$  but also the resulting CNPMLE change as well. Hence we search the optimal  $\beta$  by enumerating the possible solutions.

## 5. Numerical Examples with Reliability Application

### 5.1. Simulation experiments

First, we evaluate the estimation performance of our CNPMLE in Monte Carlo simulation. The failure data under general repair are generated by a TRP where the trend function and the failure rate are given by the power law type with  $\Lambda(t) = 100 \times t^2$  and the Weibull distribution with  $r(x) = 3 \times x^2$ . In Figure 1, we plot four representative sample paths; DS1, DS2, DS3 and DS4, where DS1 fits best the mean value function of TRP, DS2 and DS3 are located over and under the mean value function, respectively. DS4 shows an S-shaped curve around the mean value function. In these typical cases, we apply both the parametric and nonparametric estimation methods. We assume eleven statistical models; NHPP with power law type intensity function (P-NHPP), RP with gamma renewal distribution (G-RP), RP with Weibull renewal distribution (W-RP), NHGP with power law trend function (P-NHGP), NHPLP with power law trend function (P-NHPLP; real model), NHGP with decreasing trend function (D-NHGP), NHGP with increasing trend function (I-NHGP), NHPLP with decreasing trend function (D-NHPLP), NHPLP with increasing trend function (I-NHPLP), TRP with power law trend function and DFR renewal distribution (D-TRP) and TRP with power law trend function and IFR renewal distribution (I-TRP). We confirm again the interrelation among all the statistical models; NHGP, NHPLP and TRP include NHPP and RP as special cases. P-NHGP and P-NHPLP reduce to P-NHPP when the shape parameter of gamma distribution or Weibull distribution equals to 1. P-NHGP and P-NHPLP also reduce to G-RP and W-RP, respectively, when all the parameters of power law trend function equal to 1s.

Table 1 presents the goodness-of-fit performance of respective statistical models based on the maximum log-likelihood (MLL) and mean squared error (MSE). We see that P-NHGP and P-NHPLP show the better goodness-of-fit performance in the sense of maximization of MLL and minimization of MSE, compared with two RPs. Focusing on the results of NHGP, NHPLP and TRP with CNPMLE, it can be found that each statistical model with increasing CNPMLE gives the higher MLL than decreasing CNPMLE. This is because we assume the model parameters of real model to be increasing trend function and increasing failure rate function. By comparing I-NHGP, I-NHPLP and I-TRP with P-NHGP and P-NHPLP, it is seen that the three nonparametric models tend to take higher values with respect to MLL. We calculate the MLL with I-TRP by assuming  $\lambda_n = \lambda_{n-1}$  to get the pessimistic result. However, I-TRP shows the largest MLL among all statistical models with all data sets.

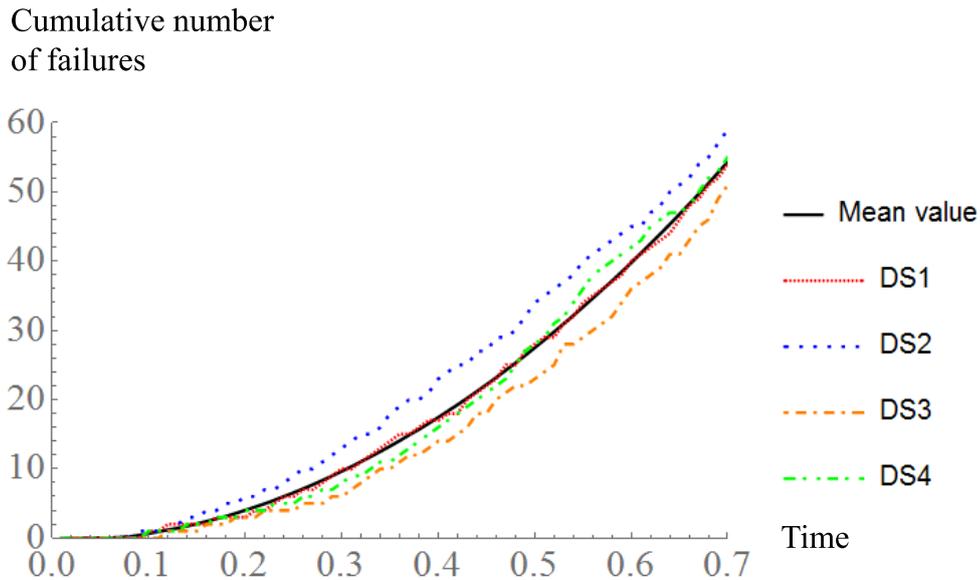


Figure 1: Simulation data

We evaluate the respective statistical models in terms of moment match and calculate the relative error on sample moments (mean, variance, skewness and kurtosis) between P-NHPLP (real) and the other statistical models in Tables 2-5. The mean, variance, skewness and kurtosis of the cumulative number of failures are numerically calculated by the Monte Carlo simulation, where the sample size in each calculation is 2000. Let  $\hat{\eta}$  and  $\eta^*$  be the sample moments derived from each statistical model and the sample moments with the real model (P-NHPLP), respectively. Since the underlying TRP is nonhomogeneous in time, we observe the sample moments at each time point; 25%, 50%, 75% and 100% point of whole data, and calculate the (percentile) relative error by

$$RE(\%) = \frac{|\hat{\eta} - \eta^*|}{\eta^*} \times 100. \tag{5.1}$$

In Tables 2-5, it can be seen that P-NHPP provides the relatively small  $RE$  with respect to mean in several cases. However, it is checked that remarkably large error on variance is observed in P-NHPP in same cases. Also, P-NHGP, P-NHPLP and I-TRP tend to give the relatively smaller  $RE$  than the other statistical models. Looking at the results on the nonparametric estimation methods, both of I-NHGP and I-NHPLP related with I-TRP tend to take values close to the real moment  $\eta^*$  in many cases, compared to the same class with decreasing failure rate function or trend function. Furthermore, it can be found that the best statistical models which minimize the relative error depend on the shape of data. More precisely, I-NHGP for DS1 and DS3, I-TRP for DS2 and P-NHPLP for DS4 provide the better performances in terms of small relative error on higher moments.

### 5.2. Real data analysis

Next we give a field data analysis based on the failure occurrence data of a diesel engine [32, 41], where the number of failures is  $n = 71$  and the mean time between failures (MTBF) is  $= 0.359$  (hr) if this data is regarded as i.i.d. samples. Here we apply eight statistical models including P-NHGP and P-NHPLP which can be classified into TRP. Figure 2 illustrates

Table 1: Goodness-of-fit performance in simulation experiments

(a) MLL					(b) MSE				
Model	DS1	DS2	DS3	DS4	Model	DS1	DS2	DS3	DS4
P-NHPP	196.33	216.27	191.66	202.49	P-NHPP	0.08	0.14	0.07	0.16
G-RP	191.10	218.56	178.30	194.45	G-RP	4.30	4.71	4.13	4.38
W-RP	187.52	212.50	176.47	191.26	W-RP	4.32	4.73	4.13	4.38
P-NHGP	225.83	252.71	215.41	232.02	P-NHGP	2.31	2.73	1.98	2.30
P-NHPLP	227.43	253.23	217.52	232.51	P-NHPLP	2.33	2.75	2.04	2.34
D-NHGP	191.10	218.56	178.30	194.45	D-NHGP	4.31	4.71	4.12	4.37
I-NHGP	225.86	254.44	215.40	233.72	I-NHGP	5.54	5.85	5.53	5.72
D-NHPLP	187.52	212.50	176.47	191.26	D-NHPLP	4.33	4.74	4.14	4.39
I-NHPLP	232.11	259.19	221.92	237.54	I-NHPLP	5.50	5.82	5.46	5.65
D-TRP	196.33	216.27	191.66	202.49	D-TRP	3.94	4.28	3.26	4.02
I-TRP	253.57	270.69	245.86	259.09	I-TRP	2.48	2.93	2.19	2.31

Table 2: Estimating mean value function of TRP

(a) DS1					(b) DS2				
Model	25%	50%	75%	100%	Model	25%	50%	75%	100%
P-NHPP	63.34%	27.88%	18.31%	12.63%	P-NHPP	41.44%	31.33%	19.98%	10.04%
G-RP	501.28%	161.11%	71.31%	27.93%	G-RP	433.74%	131.21%	51.30%	12.78%
W-RP	497.78%	158.56%	69.57%	26.41%	W-RP	433.01%	129.46%	50.00%	11.65%
P-NHGP	63.37%	31.54%	20.02%	12.43%	P-NHGP	49.22%	28.42%	17.77%	9.83%
P-NHPLP	65.56%	32.45%	20.67%	12.88%	P-NHPLP	38.25%	22.82%	14.53%	8.57%
D-NHGP	503.02%	162.71%	72.31%	28.23%	D-NHGP	435.52%	131.01%	51.10%	12.49%
I-NHGP	34.22%	29.72%	22.27%	10.77%	I-NHGP	42.28%	34.43%	19.92%	12.76%
D-NHPLP	495.56%	156.95%	68.66%	25.63%	D-NHPLP	432.55%	129.42%	50.10%	11.95%
I-NHPLP	48.73%	35.04%	22.22%	9.26%	I-NHPLP	31.85%	25.89%	14.18%	11.83%
D-TRP	153.44%	99.67%	86.23%	77.51%	D-TRP	4.25%	13.63%	34.17%	50.33%
I-TRP	124.44%	54.32%	30.20%	15.67%	I-TRP	6.09%	6.74%	5.66%	5.68%

(c) DS3					(d) DS4				
Model	25%	50%	75%	100%	Model	25%	50%	75%	100%
P-NHPP	7.24%	3.76%	2.34%	1.09%	P-NHPP	6.16%	9.51%	4.68%	0.63%
G-RP	454.96%	140.18%	56.89%	16.60%	G-RP	461.89%	143.49%	60.29%	19.32%
W-RP	439.10%	135.00%	54.98%	15.64%	W-RP	462.42%	141.96%	58.42%	18.00%
P-NHGP	2.17%	0.21%	0.57%	0.97%	P-NHGP	12.65%	5.98%	2.38%	0.03%
P-NHPLP	7.05%	1.90%	0.65%	0.49%	P-NHPLP	2.32%	0.87%	0.29%	0.73%
D-NHGP	450.74%	139.49%	57.26%	16.99%	D-NHGP	463.70%	145.35%	60.74%	19.50%
I-NHGP	2.00%	3.16%	2.97%	2.24%	I-NHGP	7.44%	18.15%	6.47%	1.28%
D-NHPLP	446.20%	136.13%	55.21%	15.66%	D-NHPLP	460.95%	143.14%	59.08%	18.44%
I-NHPLP	8.11%	4.18%	0.05%	2.80%	I-NHPLP	7.03%	10.60%	0.37%	2.56%
D-TRP	83.00%	66.10%	67.89%	70.24%	D-TRP	61.58%	60.08%	68.77%	77.00%
I-TRP	46.75%	18.89%	8.45%	1.79%	I-TRP	8.04%	3.64%	1.26%	0.25%

Table 3: Estimating variance of TRP

(a) DS1					(b) DS2				
Model	25%	50%	75%	100%	Model	25%	50%	75%	100%
P-NHPP	432.83%	553.86%	589.00%	630.23%	P-NHPP	366.24%	514.75%	572.52%	633.55%
G-RP	1297.48%	723.84%	436.76%	319.13%	G-RP	1430.51%	788.42%	502.54%	372.64%
W-RP	1650.36%	936.21%	600.81%	471.19%	W-RP	1850.15%	1035.58%	698.71%	528.13%
P-NHGP	26.08%	19.50%	18.24%	20.01%	P-NHGP	8.47%	16.29%	17.82%	29.48%
P-NHPLP	13.47%	9.29%	9.29%	8.90%	P-NHPLP	11.88%	10.13%	13.07%	16.32%
D-NHGP	1288.52%	731.40%	472.51%	333.72%	D-NHGP	1531.72%	813.29%	528.35%	390.76%
I-NHGP	3.03%	2.31%	1.52%	7.58%	I-NHGP	0.72%	15.71%	5.27%	3.45%
D-NHPLP	1702.88%	927.20%	603.26%	455.40%	D-NHPLP	1915.68%	1076.22%	682.46%	518.79%
I-NHPLP	9.34%	19.98%	22.43%	25.66%	I-NHPLP	9.41%	24.69%	21.93%	12.11%
D-TRP	810.01%	869.78%	918.24%	942.96%	D-TRP	708.38%	903.90%	952.73%	1001.62%
I-TRP	27.73%	26.85%	13.98%	9.24%	I-TRP	10.14%	4.24%	1.90%	5.10%

(c) DS3					(d) DS4				
Model	25%	50%	75%	100%	Model	25%	50%	75%	100%
P-NHPP	711.54%	768.77%	734.75%	731.51%	P-NHPP	190.95%	366.51%	464.60%	564.14%
G-RP	1086.34%	609.90%	374.31%	265.97%	G-RP	1717.25%	974.84%	638.03%	457.75%
W-RP	1610.29%	918.02%	560.99%	404.77%	W-RP	2047.43%	1172.77%	815.20%	584.41%
P-NHGP	46.25%	44.10%	27.14%	24.04%	P-NHGP	12.17%	15.52%	29.29%	52.15%
P-NHPLP	38.00%	26.77%	9.93%	8.41%	P-NHPLP	11.91%	1.42%	4.80%	14.85%
D-NHGP	1117.15%	602.94%	375.35%	271.90%	D-NHGP	1605.10%	899.52%	561.37%	429.79%
I-NHGP	9.03%	14.05%	4.52%	0.39%	I-NHGP	14.06%	13.92%	2.67%	13.74%
D-NHPLP	1646.20%	843.45%	532.00%	381.25%	D-NHPLP	2049.86%	1168.74%	739.74%	556.99%
I-NHPLP	0.64%	11.63%	18.54%	27.57%	I-NHPLP	30.84%	35.64%	28.60%	25.31%
D-TRP	1088.69%	1029.96%	982.73%	952.32%	D-TRP	480.08%	566.07%	644.83%	790.30%
I-TRP	91.44%	48.50%	25.74%	12.96%	I-TRP	28.78%	17.57%	14.99%	12.40%

Table 4: Estimating skewness of TRP

(a) DS1					(b) DS2				
Model	25%	50%	75%	100%	Model	25%	50%	75%	100%
P-NHPP	151.49%	0.85%	29.70%	22.21%	P-NHPP	110.47%	43.67%	28.36%	9.68%
G-RP	18.79%	72.19%	54.99%	93.62%	G-RP	5.50%	58.21%	41.50%	32.54%
W-RP	33.75%	46.50%	42.04%	13.55%	W-RP	22.47%	60.94%	33.11%	45.76%
P-NHGP	34.33%	110.71%	44.24%	35.02%	P-NHGP	39.20%	55.71%	42.35%	92.36%
P-NHPLP	15.26%	44.31%	42.23%	0.64%	P-NHPLP	22.02%	9.76%	23.54%	2.98%
D-NHGP	21.40%	55.56%	14.43%	13.78%	D-NHGP	6.72%	49.22%	38.92%	36.72%
I-NHGP	23.51%	86.68%	74.17%	50.47%	I-NHGP	29.42%	40.31%	53.32%	72.56%
D-NHPLP	0.23%	64.65%	23.61%	61.47%	D-NHPLP	4.46%	65.75%	21.12%	20.54%
I-NHPLP	73.24%	40.69%	58.23%	109.18%	I-NHPLP	7.26%	45.13%	1.77%	30.31%
D-TRP	231.33%	25.09%	152.37%	136.38%	D-TRP	130.92%	0.78%	114.19%	34.85%
I-TRP	1.91%	6.74%	107.27%	8.75%	I-TRP	3.37%	38.78%	48.78%	8.93%

(c) DS3					(d) DS4				
Model	25%	50%	75%	100%	Model	25%	50%	75%	100%
P-NHPP	184.03%	5.79%	31.40%	63.47%	P-NHPP	159.79%	3.88%	99.57%	71.24%
G-RP	14.15%	51.05%	40.52%	26.68%	G-RP	37.79%	72.95%	11.13%	21.03%
W-RP	21.96%	44.68%	54.44%	16.42%	W-RP	6.76%	39.79%	36.55%	15.54%
P-NHGP	53.06%	63.33%	30.61%	50.56%	P-NHGP	33.54%	68.48%	65.73%	43.76%
P-NHPLP	98.61%	30.45%	115.29%	93.82%	P-NHPLP	33.75%	43.37%	42.11%	14.82%
D-NHGP	40.45%	89.09%	18.79%	34.39%	D-NHGP	14.34%	76.95%	9.10%	3.92%
I-NHGP	12.61%	90.17%	99.36%	113.50%	I-NHGP	66.79%	50.00%	33.61%	90.79%
D-NHPLP	37.61%	40.90%	17.87%	33.61%	D-NHPLP	35.27%	30.49%	13.22%	49.36%
I-NHPLP	142.19%	52.17%	22.01%	28.64%	I-NHPLP	40.68%	5.62%	70.69%	23.08%
D-TRP	292.62%	52.45%	94.56%	35.74%	D-TRP	295.29%	61.42%	110.58%	46.87%
I-TRP	108.91%	18.00%	10.04%	7.79%	I-TRP	23.68%	20.54%	27.07%	5.10%

Table 5: Estimating kurtosis of TRP

(a) DS1					(b) DS2				
Model	25%	50%	75%	100%	Model	25%	50%	75%	100%
P-NHPP	8.92%	2.71%	1.89%	2.51%	P-NHPP	0.50%	4.24%	3.19%	3.91%
G-RP	2.81%	1.89%	0.21%	1.28%	G-RP	7.73%	5.48%	1.62%	5.45%
W-RP	1.84%	6.30%	1.93%	4.22%	W-RP	11.85%	7.90%	3.43%	2.88%
P-NHGP	10.70%	2.52%	4.98%	1.96%	P-NHGP	3.60%	2.73%	4.24%	5.78%
P-NHPLP	3.72%	5.36%	6.15%	2.91%	P-NHPLP	3.74%	5.11%	1.01%	0.16%
D-NHGP	7.46%	2.59%	3.18%	0.92%	D-NHGP	9.05%	7.36%	5.12%	4.50%
I-NHGP	0.43%	2.11%	0.77%	1.61%	I-NHGP	15.89%	5.73%	0.27%	6.33%
D-NHPLP	7.16%	10.03%	2.51%	5.27%	D-NHPLP	7.64%	9.63%	4.48%	3.25%
I-NHPLP	10.73%	8.06%	8.02%	1.03%	I-NHPLP	3.26%	8.86%	5.61%	1.44%
D-TRP	22.86%	1.71%	3.12%	2.84%	D-TRP	0.86%	4.54%	5.30%	4.62%
I-TRP	11.79%	2.78%	0.18%	0.47%	I-TRP	13.61%	1.98%	7.07%	0.66%

(c) DS3					(d) DS4				
Model	25%	50%	75%	100%	Model	25%	50%	75%	100%
P-NHPP	0.61%	0.48%	3.96%	3.55%	P-NHPP	1.34%	1.46%	6.14%	3.46%
G-RP	8.77%	10.18%	9.19%	9.24%	G-RP	12.60%	7.55%	2.21%	0.94%
W-RP	0.51%	12.59%	1.49%	2.20%	W-RP	6.01%	4.77%	1.99%	2.02%
P-NHGP	3.60%	3.89%	3.44%	2.35%	P-NHGP	11.66%	5.38%	6.87%	8.40%
P-NHPLP	15.08%	5.38%	3.36%	2.39%	P-NHPLP	1.55%	8.02%	2.95%	0.21%
D-NHGP	8.41%	6.28%	1.01%	5.65%	D-NHGP	4.04%	6.19%	1.86%	4.82%
I-NHGP	0.55%	3.50%	4.21%	0.79%	I-NHGP	12.76%	3.95%	2.78%	4.44%
D-NHPLP	3.01%	4.97%	3.56%	6.64%	D-NHPLP	1.52%	5.82%	4.16%	1.00%
I-NHPLP	26.67%	9.98%	0.09%	1.21%	I-NHPLP	5.55%	1.90%	0.70%	5.47%
D-TRP	40.56%	0.55%	2.91%	2.51%	D-TRP	50.74%	10.80%	2.06%	0.96%
I-TRP	5.85%	0.36%	3.99%	8.82%	I-TRP	11.00%	7.84%	4.05%	3.64%

the estimation results of respective statistical models, where the step function in the figure denotes the real data. From this observation, it can be shown that NHGP and NHPLP with decreasing trend function result big differences between the mean value functions and the real data. On the other hand, TRP with decreasing failure rate function gives relatively closed value to the real data compared with the NHGP and NHPLP with decreasing trend function. Also, I-NHGP and I-NHPLP tend to give the lower bounds of real data. These results mean that the decreasing or increasing property of trend function influences the shape of mean value rather than the monotone property of failure rate function. By shifting the observation point from 25% to 100%, we calculate the MLL and MSE in Table 6. From this table, we can find that I-NHGP and I-NHPLP show the better goodness-of-fit performance, compared to decreasing case of each statistical model, except for the MSE at 25% observation point. In addition, I-TRP tends to give the largest MLL among all the statistical models and shows relatively small MSE, especially at 75% and 100% observation points.

Next, we consider a periodic replacement problem [41, 43] based on the TRP. The long-run average cost per unit time is formulated by

$$C(t) = \frac{c_1 E[N(t)] + c_2}{t}, \tag{5.2}$$

where  $c_1 (> 0)$  and  $c_2 (> 0)$  represent the fixed costs of the general repair and periodic replacement, respectively. We obtain the optimal periodic replacement time  $t^*$  which minimizes Equation (5.2) under a general repair. Since P-NHPP and RP represent the minimal repair and the major (perfect) repair, respectively, it is evident that our problem with TRP can express several kinds of repair activities. Table 7 presents the optimal periodic replacement times and their associated minimum expected costs with eight statistical models at

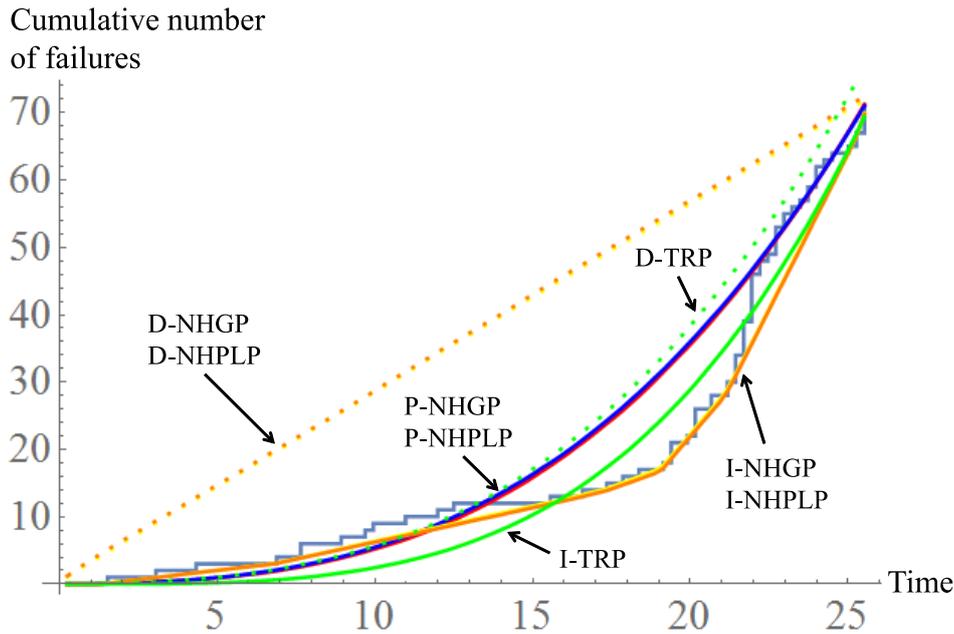


Figure 2: Estimating of mean value functions of TRPs

Table 6: Goodness-of-fit performance in real data analysis

(a) MLL					(b) MSE				
Model	25%	50%	75%	100%	Model	25%	50%	75%	100%
P-NHGP	-16.313	-5.781	19.467	33.187	P-NHGP	0.214	0.828	1.160	0.774
P-NHPLP	-16.640	-5.862	19.785	33.408	P-NHPLP	0.208	0.792	1.129	0.800
D-NHGP	-17.238	-12.286	6.763	16.293	D-NHGP	0.263	1.542	2.324	2.565
I-NHGP	-15.167	7.215	33.894	44.718	I-NHGP	0.401	0.325	0.411	0.654
D-NHPLP	-17.314	-12.828	7.217	17.776	D-NHPLP	0.273	1.410	2.308	2.608
I-NHPLP	-14.553	7.206	33.827	44.575	I-NHPLP	0.380	0.280	0.442	0.677
D-TRP	-18.453	-4.746	22.693	36.763	D-TRP	0.661	1.332	1.810	1.025
I-TRP	8.978	4.872	32.538	55.916	I-TRP	0.416	1.512	0.577	0.574

100% observation point, where TRPs with decreasing case are omitted because of their low goodness-of-fit performances, and the cost parameters are given by  $c_1 = 1, c_2 = 1$ . From this result, it can be seen that P-NHPP and two RPs give the shortest and longest optimal periodic replacement times, respectively, since the minimal repair denotes an emergency repair which does not affect the failure rate and the major repair denotes essentially a replacement of failure component by a new one. Since the optimal periodic replacement time with TRPs relates the general repair, it ranges between the respective replacement times for P-NHPP and RPs. On the other hand, the minimum expected cost with I-TRP is the smallest cost among all the statistical models. Then we can know that TRP with increasing failure rate function gives the more optimistic estimation result than the other statistical models.

Finally, we perform the sensitivity analysis in the periodic replacement problem in Table 8. For varying model parameters of P-NHGP and P-NHPLP, we investigate the change of optimal periodic replacement times and the corresponding minimal expected costs where the cost parameters are fixed as  $c_1 = 1, c_2 = 1$ . As the shape parameter  $\kappa$  ( $b$ ) of gamma

(Weibull) renewal distribution approaches to 1, P-NHGP and P-NHPLP get closer to P-NHPP. On the other hand, as the shape parameter  $\kappa$  ( $b$ ) of power law trend function approaches to 1, P-NHGP and P-NHPLP get closer to G-RP and W-RP as well. Looking at the optimal periodic replacement time with P-NHPLP, it can be found that the resulting optimal replacement time gets closer monotonously to those of P-NHPP and W-RP as the parameter changes. On the other hand, focusing on the results with P-NHGP, we can see that the resulting minimal expected cost approaches gradually to the results of P-NHPP and G-RP.

Table 7: Optimal periodic replacement times and their associated minimal expected costs

	$c_1 = 1 / c_2 = 1$	
Model	Time	Cost
P-NHPP	4.44	0.35
G-RP	24.05	2.83
W-RP	25.49	2.89
P-NHGP	4.56	0.38
P-NHPLP	4.70	0.40
I-NHGP	6.83	0.60
I-NHPLP	6.67	0.60
I-TRP	5.80	0.23

Table 8: Sensitivity analysis of model parameters in P-NHGP and P-NHPLP

	P-NHPLP		P-NHGP			P-NHPLP		P-NHGP	
$b$ ( $\kappa$ )	Time	Cost	Time	Cost	$\beta$	Time	Cost	Time	Cost
0.2	10.56	0.28	6.63	0.43	1.8	2.92	0.97	3.01	0.93
0.3	6.99	0.36	6.04	0.42	1.7	2.95	1.08	2.79	1.06
0.4	5.50	0.39	5.18	0.41	1.6	3.06	1.26	2.94	1.20
0.5	5.10	0.42	4.59	0.39	1.5	3.09	1.45	2.50	1.41
0.6	5.08	0.41	4.85	0.37	1.4	3.26	1.68	2.68	1.60
0.7	4.77	0.41	4.80	0.38	1.3	3.44	1.98	2.69	1.90
0.8	4.61	0.38	4.43	0.37	1.2	3.76	2.31	3.60	2.29
0.9	4.49	0.38	4.34	0.35	1.1	6.74	2.70	5.08	2.61
1.0	4.33	0.36	4.50	0.35	1	25.48	2.89	25.45	2.84

## 6. Conclusion

In this paper, we have proposed another nonparametric estimation algorithm for trend renewal processes and shown that our constrained nonparametric maximum likelihood estimator for a given trend function gave the higher goodness-of-fit performance in many cases in terms of maximum likelihood, mean squared error and moment matching. Furthermore, we have applied our statistical estimation method to a periodic replacement problem as well as the life data analysis of a repairable system. Our method developed in this paper is a dual approach for the nonparametric monotone maximum likelihood estimator by Heggland

and Lindqvist [21] and will complement their result under the assumption that the trend function is unknown. This is an achievement to derive one of the most useful nonparametric statistical inference tools for general stochastic point processes such as TRPs. In future, we plan to develop a CNPMLE for the most complex case where both of trend function and the failure rate function are unknown. Also, we will consider a kernel-based approach (see Saito and Dohi [43]) for the same nonparametric inference of TRPs.

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