

AN APPROXIMATE BARRIER OPTION MODEL FOR VALUING EXECUTIVE STOCK OPTIONS

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Abstract A continuous-time barrier option model is developed for valuing executive stock options (ESOs), in which early exercise takes place whenever the underlying stock price reaches a certain upper barrier after vesting. We analyze the ESO value and the ESO exercise time to obtain their solutions in simple forms, which are consistent with principal features of early exercise, delayed vesting and random exit. For the perpetual case, these solutions are given in explicit forms and shown to be exact in the Black-Scholes-Merton formulation. Using an endogenous approximation for the barrier level, we numerically compare our approximation for the ESO value with a benchmark result generated by a binomial-tree model and the quadratic approximation previously established. From numerical comparisons for some particular cases, we see that our approximations always underestimate the benchmark results and the absolute values of the relative percentage errors are less than 1% for all cases, whereas the quadratic approximations overestimate the benchmarks and the relative percentage errors are less than about 2%.

Keywords: Finance, executive stock options, ESO, endogenous approximation, valuation, barrier option

1. Introduction

Executive (or employee) stock options (ESOs) have become increasingly popular and currently constitute a certain fraction of total compensation expense of many firms. ESOs are call options that give the option holder the right to buy their firm's stock for a fixed strike price during a specified period of time. Clearly, the exercise of ESOs triggers a dilution of the claims of the firm's existing shareholders, since the firm issues new stocks to the ESO holders. The Financial Accounting Standards Board (FASB) issued in 2004 a revised version of Statement of Financial Accounting Standard No. 123, Share-Based Payments (SFAS 123R), which requires firms to estimate and report the *fair value* of ESOs at the grant date. ESO valuation is now becoming an important issue in many countries, which inspires us to create a reasonable valuation method for ESOs.

ESOs have features different from ordinary market-traded options. While ordinary options usually mature within one year, ESOs have maturity over many years, typically, it is set equal to ten years. Also, ESOs are usually granted at-the-money, namely, its strike price is set equal to the current stock price. During the beginning part of the option's life (called a vesting period), ESO holders cannot exercise their options and must forfeit the options on leaving the firm. Typically, the length of vesting period is two or three years after the grant date. After the vesting date, ESO holders can exercise the options at any time before maturity date, i.e., ESOs are of American-style. The most significant difference between ESOs and traded options would be that ESO holders cannot sell or otherwise transfer them. An ESO holder leaving the firm is then forced to choose between forfeiting or exercising the options soon after his exit. The lack of transferability implies that ESO holders cannot

hedge their positions, and so that their personal valuations depend on their risk preferences and endowments. Thus the non-transferability of ESOs may be realized in mathematical models by maximizing a utility function of ESO holders. Through an empirical analysis using data on ESO exercises from 40 firms, however, Carpenter [6] showed that a simple American option pricing model performs well as an elaborate utility-maximizing model. Hence, this paper considers ESOs with three principal features, namely, *early exercise*, *delayed vesting* and *employment termination*. Additional features such as resetting/reloading provisions and multiple exercising rights are not considered here.

No doubt, modeling based on American options is a natural choice for incorporating the early exercise feature into ESO valuation models. However, it has been known that even a vanilla American call option written on a dividend-paying stock has no closed-form valuation formula. Hence, it is inevitable to use a binomial-tree (i.e., discrete-time) model in ESO valuation with the American framework, except for the perpetual case with infinite maturity. Binomial-tree or lattice models, dating back to Cox et al. [10], have been extensively used by many researchers not only as a tool for empirical studies [3, 6, 13] but also as a benchmark for testing their models [1, 17, 25, 27]. In addition, FASB authorized a binomial-tree model as an acceptable valuation method in SFAS 123R. For continuous-time models, however, there have been relatively few studies in the American framework, due to the analytical difficulty. Sircar and Xiong [27] developed a general model for perpetual ESOs with additional features of resetting and reloading. Recently, Leung and Sircar [22] formulated the ESO valuation problem as a chain of nonlinear free-boundary problems of reaction-diffusion type, combining effects of risk aversion, trading and hedging constraints, and job termination risk. They analyzed PDE problems numerically using the implicit finite-difference methods to obtain the exercise boundary and the ESO value. Kimura [17] has developed a continuous-time model to obtain an explicit valuation formula, by applying the quadratic approximation originally established for valuing American options [2].

As opposed to the American framework, there exists a different framework based on European options. Of course, some artificial invention is necessarily required to realize the early exercise feature in the European framework. A very primitive idea is to put forward maturity date in the Black-Scholes model in such a way that maturity is replaced with the expected time to ESO exercise. Clearly, this primitive method is far from accurate [13, 15] and it would be ruled out from accounting standards. As another idea, Carr and Linetsky [8] developed an intensity-based model in the European framework to realize the early exercise feature as well as forfeiture due to voluntary or involuntary employment termination; see Jennergren and Näslund [15] for early work. This model is primarily aimed to capture the fact that ESO holders may exercise earlier than standard theory predicts, but neglecting the ESO holder's own exercising policy.

An alternative idea in the European framework is to model the early exercise behavior of ESO holders by assuming the exercise takes place whenever the stock price reaches a certain upper barrier. The barrier option models have been developed by Hull and White [14], Raupach [25] and Cvitanić et al. [11]. Hull and White [14] assume a flat barrier in a binomial-tree model, whereas an exponentially decaying/improving barrier in a continuous-time model is assumed in Raupach [25] and Cvitanić et al. [11]. The barrier option models have an advantage of easy handling, but they lack an internal mechanism of determining the optimal barrier level, unlike the American framework. Hence, we have to specify in advance the early exercise barrier *exogenously*, which means that shape parameters have to be estimated from actual data of exercise levels. The exponential early exercise barriers of the continuous-time models in [11, 25] are convex functions of time to maturity, being

inconsistent with the concavity observed in models of the American framework; see, e.g., Figure 5 of Ammann and Seiz [1]. Moreover, Raupach [25] showed via numerical experiments that the decaying/improving rate is a weak value driver, compared to other parameters. From these results, we may consider that a flat (i.e., constant) barrier would be sufficient for valuation if it works well.

With respect to the modeling of employment termination, there is a significant difference between Raupach [25] and Cvitanić et al. [11]: Raupach analyzed the ESO value assuming no exit before maturity and then obtain the unconditional value using a stopping time argument, while Cvitanić et al. adopted the intensity-based framework in Carr and Linetsky [8]. The different modeling yields that Raupach's result has a form of a two-dimensional integral and the result of Cvitanić et al. is represented by a number of lengthy formulas extending to nine pages [11, pp. 702–710].

This paper has the threefold purposes as follows:

1. to develop an explicit formula for the ESO value;
2. to obtain the ESO exercise time distribution and its mean; and
3. to provide ESO holders with a simple rule of making exercise decision quickly,

all of which are realized by a continuous-time barrier option model with an *endogenously* specified flat barrier, where we adopt Raupach's modeling of employment termination. The formula for the ESO value (Theorem 1) is much more compact than those in Raupach and Cvitanić et al., and explicit formulas for the ESO exercise time (Theorems 3 through 5) are new results. Endogenous specification of the barrier level is based on simple heuristics, but it is shown from numerical results for some particular cases that the associated approximation for the ESO value performs well, i.e., the approximate ESO value is very close to a theoretical upper limit of the barrier-option model. The absolute values of the relative percentage errors based on benchmark results by an American binomial-tree model are less than 1% for all cases (Tables 1 and 2).

The rest of this paper is organized as follows: In Section 2, we formulate the underlying stock process in the Black-Scholes-Merton economy, and then we show that vested ESOs can be adequately modeled by an American call option written on that process. From the shapes of early exercise boundaries, we observe that the early exercise behavior of the American option model can be effectively substituted for a boundary behavior of a European barrier option model. This observation enables us to analyze the ESO value in Section 3, obtaining a simple valuation formula consistent with the principal ESO features. In Section 4, we also obtain explicit formulas for the time of ESO exercise. For both of these quantities, we derive closed-form formulas for the perpetual case, which are exact in the Black-Scholes-Merton formulation. In Section 5, we propose a simple approximation for the barrier level actually used in these formulas. Section 6 is devoted to numerical examinations of our formulas. Finally, Section 7 concludes the paper with some remarks on future research.

2. Black-Scholes-Merton Formulation

Suppose an economy with finite time period $[0, T]$, a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t \in [0, T]}$. A standard Brownian motion process $W \equiv (W_t)_{t \in [0, T]}$ is defined on (Ω, \mathcal{F}) and takes values in \mathbb{R} . The filtration is the natural filtration generated by W and $\mathcal{F}_T = \mathcal{F}$. Let $(S_t)_{t \in [0, T]}$ be the price process of the underlying stock. For S_0 given, assume that $(S_t)_{t \in [0, T]}$ is a geometric Brownian motion process

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t, \quad t \in [0, T], \quad (2.1)$$

where $r > 0$ is the risk-free rate of interest, $\delta \geq 0$ the continuous dividend rate, and $\sigma > 0$ the volatility coefficient of stock returns. Assume that all of the coefficients (r, δ, σ) are constant. The stock price process $(S_t)_{t \in [0, T]}$ is represented under the equivalent martingale (i.e., risk-neutral) measure \mathbb{P} , which indicates that the stock has mean rate of return r , and the process W is a \mathbb{P} -Brownian motion.

Consider an ESO written on the stock price process $(S_t)_{t \in [0, T]}$, which has maturity date T and strike price K . Let $t = 0$ and $t = T_1$ ($\in (0, T)$) respectively denote the grant and vesting dates of the ESO. Since the ESO can be exercised at any time during $[T_1, T]$, it is adequate to formulate the vested ESO as an ordinary American call option (or its extension) written on a dividend-paying stock in this time interval. For the sake of a clear argument, assume for a while that no exit from the firm occurs before maturity. Let $C(S_t, t) \equiv C(S_t, t; T, K)$ denote the value of the American vanilla call option at time $t \in [T_1, T]$ with maturity date T and strike price K . In the absence of arbitrage opportunities, the value $C(S_t, t)$ is a solution of an *optimal stopping problem*

$$C(S_t, t) = \operatorname{ess\,sup}_{\tau_e \in [t, T]} \mathbb{E} \left[e^{-r(\tau_e - t)} (S_{\tau_e} - K)^+ \mid \mathcal{F}_t \right], \quad (2.2)$$

for $t \in [T_1, T]$, where $(x)^+ = \max(x, 0)$ for $x \in \mathbb{R}$, τ_e is a stopping time of the filtration $(\mathcal{F}_t)_{t \geq 0}$, and the conditional expectation is calculated under the measure \mathbb{P} . The random variable $\tau_e^* \in [t, T]$ is called an *optimal stopping time* if it gives the supremum value of the right-hand side of (2.2). Let $\mathcal{D}_0 = \{(S_t, t) \in \mathbb{R}_+ \times [0, T]\}$ denote the whole domain and let $\mathcal{D} = \{(S_t, t) \in \mathbb{R}_+ \times [T_1, T]\}$ denote the subdomain of \mathcal{D}_0 after the vesting date. Solving the optimal stopping problem (2.2) is equivalent to finding the points $(S_t, t) \in \mathcal{D}$ for which early exercise of the ESO is optimal. Let \mathcal{C} and \mathcal{E} denote the *continuation region* and *exercise region*, respectively. Clearly, the continuation region \mathcal{C} is the complement of \mathcal{E} in \mathcal{D} , and the boundary that separates \mathcal{E} from \mathcal{C} is referred to as an *early exercise boundary*, which is defined by

$$\bar{S}_t = \inf \{ S_t \in \mathbb{R}_+ \mid C(S_t, t) = (S_t - K)^+ \}, \quad t \in [T_1, T]. \quad (2.3)$$

Since $C(S_t, t)$ is nondecreasing in S_t , the early exercise boundary $(\bar{S}_t)_{t \in [T_1, T]}$ is an *upper critical stock price*, above which it is advantageous to exercise the ESO before maturity. Also, it has been known in Kim [16] that

$$\bar{S}_T = \max \left(1, \frac{r}{\delta} \right) K. \quad (2.4)$$

Let $C_\infty(S)$ be the value of the perpetual American call option at the vesting date T_1 with strike price K and initial stock price $S_{T_1} \equiv S$, i.e., $C_\infty(S) = \lim_{T \rightarrow \infty} C(S, T_1; T, K)$. Then, from McKean [23], we have

$$C_\infty(S) = \begin{cases} \frac{\bar{S}}{\theta} \left(\frac{S}{\bar{S}} \right)^\theta, & S < \bar{S} \\ S - K, & S \geq \bar{S}, \end{cases} \quad (2.5)$$

where \bar{S} ($> K$) is the value of the perpetual early exercise boundary, given by

$$\bar{S} = \frac{\theta}{\theta - 1} K, \quad (2.6)$$

and $\theta > 1$ is a positive root of the quadratic equation

$$\frac{1}{2} \sigma^2 \theta^2 + (r - \delta - \frac{1}{2} \sigma^2) \theta - r = 0, \quad (2.7)$$

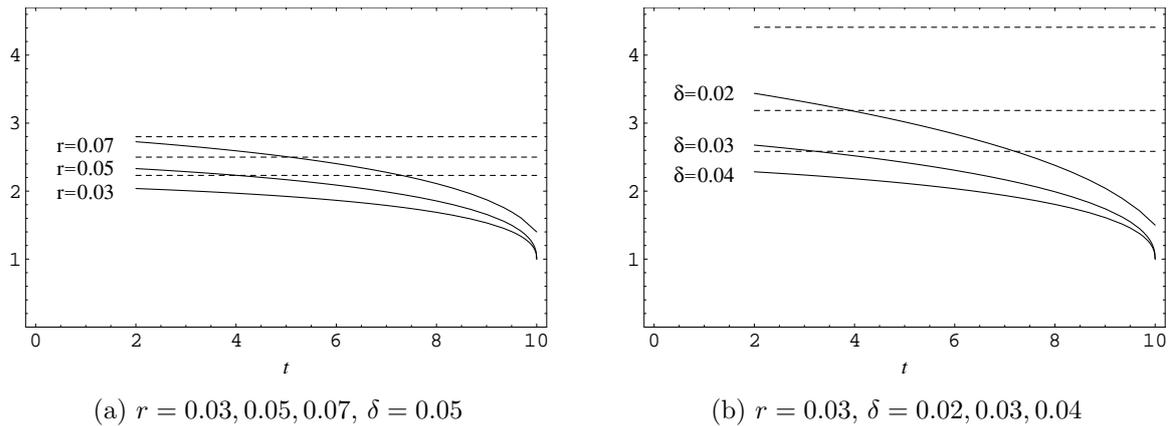


Figure 1: Normalized early exercise boundary $(\bar{S}_t/K)_{t \in [T_1, T]}$ ($T_1 = 2, T = 10, \sigma = 0.3$)

namely,

$$\theta = \frac{1}{\sigma^2} \left\{ -(r - \delta - \frac{1}{2}\sigma^2) + \sqrt{(r - \delta - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r} \right\}. \quad (2.8)$$

For some typical cases, Figure 1 illustrates the normalized early exercise boundary $(\bar{S}_t/K)_{t \in [T_1, T]}$ of vested ESOs as functions of the elapsed time. The effects of the risk-free rate r to the boundaries can be shown in Figure 1(a), while Figure 1(b) shows the effects of the dividend rate δ . To draw the boundaries, we applied the Gaver-Stehfest method of Laplace transform inversion to the American vanilla call option; see Kimura [18] for details of the algorithm. In these figures, the dashed lines represent the associated boundaries \bar{S} for the perpetual case. We can see that the terminal values at maturity are consistent with the theoretical result in (2.4). From Figure 1, we observe that the boundary value is an increasing (a decreasing) function of r (δ). We also observe that the early exercise boundaries $(\bar{S}_t)_{t \in [T_1, T]}$ for relatively large δ are almost flat at least during the initial period of the interval $[T_1, T]$. This observation partly supports the idea that the early exercise behavior of the American option model can be effectively substituted for a boundary behavior of a European barrier option model [26] with a flat *knock-out* barrier at a level between \bar{S}_T and \bar{S} . From Figure 1(b), however, it would be expected that the barrier-option approximation performs poorly as the dividend rate δ tends to zero, in particular for no-exit cases; see Section 6 for numerical examinations.

3. Valuation with a Barrier Option

3.1. An up-and-out call option

For the ESO valuation problem, Hull and White [14] have developed a binomial-tree model in the European framework, assuming the early exercise policy such that an executive exercises her/his vested ESOs when the stock price breaches a pre-specified target value, say H ($> K$). If the target value H is attained prior to maturity, then the executive receives $H - K$, otherwise (s)he receives $(S_T - K)^+$ at maturity. Let $M = H/K$ be the early exercise multiple of the strike price. In general, it is quite difficult to estimate M , because it depends on the market situations. In fact, there is no theoretical research for finding an *optimal* value of M in the literature on ESOs. Empirically, Carpenter [6, Table 1] reported that $M \approx 2.75$ on average from a sample of ESO exercises by top executives at 40 firms, whereas Huddart and Lang [13, Table 4] found that $M \approx 2.22$ on average from another sample of exercises by all employees at 8 firms. An endogenously specified approximation

for H , or equivalently for M , will be given in Section 5.

We shall derive an explicit valuation formula by using a continuous-time model of an up-and-out European call option with an upper barrier at $H (> K)$ and a rebate $R = H - K (> 0)$. Let $\Phi(x)$ for $x \in \mathbb{R}$ denote the standard normal cumulative distribution function (cdf), ϕ be its probability density function (pdf), i.e.,

$$\Phi(x) = \int_{-\infty}^x \phi(t)dt \quad \text{with} \quad \phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2},$$

and for $x, y, \tau > 0$ let

$$d_{\pm}(x, y, \tau) = \frac{\log(x/y) + (r - \delta \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Then, rearranging the results of Rich [26], we have a much simpler expression for the up-and-out call value as follows:

Lemma 1. *Let $c_H(S_t, t; T, K)$ denote the value of the up-and-out European call option at time $t \in [T_1, T]$, with strike K , barrier at $H (> K)$, rebate R and maturity T . Also, let $c(S_t, t; T, K)$ denote the value of the associated European vanilla call option. Then, we have*

$$\begin{aligned} c_H(S_t, t; T, K) &= c(S_t, t; T, K) - c(S_t, t; T, H) \\ &\quad - \left(\frac{H}{S_t}\right)^{2\alpha} \left\{ c\left(\frac{H^2}{S_t}, t; T, K\right) - c\left(\frac{H^2}{S_t}, t; T, H\right) \right\} \\ &\quad - (H - K)e^{-r(T-t)} \left\{ \Phi(d_-(S_t, H, T - t)) - \left(\frac{H}{S_t}\right)^{2\alpha} \Phi(d_-(H, S_t, T - t)) \right\} \\ &\quad + R \left\{ \left(\frac{H}{S_t}\right)^{\alpha+\beta} \Phi(-h_+(H, S_t, T - t)) + \left(\frac{H}{S_t}\right)^{\alpha-\beta} \Phi(-h_-(H, S_t, T - t)) \right\}, \end{aligned}$$

where the parameters α and β are given by

$$\alpha = \frac{1}{\sigma^2} (r - \delta - \frac{1}{2}\sigma^2), \quad \beta = \sqrt{\alpha^2 + \frac{2r}{\sigma^2}}, \tag{3.1}$$

and h_{\pm} is defined for $x, y, \tau > 0$ as

$$h_{\pm}(x, y, \tau) = \frac{\log(x/y) \pm \beta\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

Proof. First note that the European vanilla call value $c(S_t, t; T, K)$ is given by the well-known Black-Scholes formula

$$c(S_t, t; T, K) = S_t e^{-\delta(T-t)} \Phi(d_+(S_t, K, T - t)) - K e^{-r(T-t)} \Phi(d_-(S_t, K, T - t)).$$

From Equations (A12) and (35) in Rich [26], we have

$$\begin{aligned} c_H(S_t, t; T, K) &= C_1 - C_2 - \left(\frac{H}{S_t}\right)^{2\alpha} (C_3 - C_4) \\ &\quad + R \left\{ \left(\frac{H}{S_t}\right)^{\alpha+\beta} \Phi(-h_+(H, S_t, T - t)) + \left(\frac{H}{S_t}\right)^{\alpha-\beta} \Phi(-h_-(H, S_t, T - t)) \right\}, \end{aligned}$$

where

$$\begin{aligned}
C_1 &= S_t e^{-\delta(T-t)} \Phi(d_+(S_t, K, T-t)) - K e^{-r(T-t)} \Phi(d_-(S_t, K, T-t)) \\
&= c(S_t, t; T, K), \\
C_2 &= S_t e^{-\delta(T-t)} \Phi(d_+(S_t, H, T-t)) - K e^{-r(T-t)} \Phi(d_-(S_t, H, T-t)), \\
C_3 &= (H^2/S_t) e^{-\delta(T-t)} \Phi(d_+(H^2/S_t, K, T-t)) - K e^{-r(T-t)} \Phi(d_-(H^2/S_t, K, T-t)) \\
&= c(H^2/S_t, t; T, K), \\
C_4 &= (H^2/S_t) e^{-\delta(T-t)} \Phi(d_+(H, S_t, T-t)) - K e^{-r(T-t)} \Phi(d_-(H, S_t, T-t)).
\end{aligned}$$

The partial value C_2 can be rewritten as

$$\begin{aligned}
C_2 &= S_t e^{-\delta(T-t)} \Phi(d_+(S_t, H, T-t)) - H e^{-r(T-t)} \Phi(d_-(S_t, H, T-t)) \\
&\quad + (H - K) e^{-r(T-t)} \Phi(d_-(S_t, H, T-t)) \\
&= c(S_t, t; T, H) + (H - K) e^{-r(T-t)} \Phi(d_-(S_t, H, T-t)).
\end{aligned}$$

Also, the partial value C_4 can be rewritten similarly, using the relation

$$d_{\pm}(H^2/S_t, H, T-t) = d_{\pm}(H, S_t, T-t).$$

□

3.2. Employment termination

We now incorporate the random exit feature into the model. Executives may leave the firm voluntarily or involuntarily during the contract period $[0, T]$. They lose unvested ESOs if they leave the firm during the vesting period $[0, T_1]$. After vesting, however, they may leave the firm, thereby exercising or forfeiting vested ESOs. Let $V^\circ(S; T)$ denote the ESO value at grant date $t = 0$ with maturity date T and initial stock price $S_0 = S$, assuming no exit before maturity. This value is equivalent to that of a contingent claim of either receiving at the vesting date the up-and-out European call option with value $c_H(S_{T_1}, T_1; T, K)$ if $S_{T_1} < H$, or exercising it immediately to receive $S_{T_1} - K$ otherwise. Hence, by the risk-neutral valuation principle, we have

$$V^\circ(S; T) = e^{-rT_1} \mathbb{E} \left[c_H(S_{T_1}, T_1; T, K) \mathbf{1}_{\{S_{T_1} < H\}} + (S_{T_1} - K) \mathbf{1}_{\{S_{T_1} \geq H\}} \middle| \mathcal{F}_0 \right], \quad (3.2)$$

where $\mathbf{1}_A$ is the indicator function of A ($\in \mathcal{F}$).

Following Jennergren and Näslund [15], assume that exit from the firm occurs according to a Poisson process with an exogenous constant rate λ (> 0), and that the exit process is independent of the stock price process $(S_t)_{t \in [0, T]}$; cf. Carr and Linetsky [8] for more general point processes with rate dependent on the stock price. Let $V(S; T)$ denote the value of the associated ESO with exit rate λ in the barrier option model. Following Raupach [25], we have

Lemma 2. *For $0 < T_1 < T$, we have*

$$V(S; T) = e^{-\lambda T} V^\circ(S; T) + \int_{T_1}^T \lambda e^{-\lambda t} V^\circ(S; t) dt. \quad (3.3)$$

Proof. Let $\tau_H \equiv \tau_H(S) = \inf\{t \geq 0 \mid S_t \geq H, S_0 = S\}$ be the first passage time at which the stock price process S_t hits the barrier H for the first time, which will be analyzed independently in Section 4. There exists a stopping time τ_f such that for $\tau_f \leq \tau_H$ (a.s.) the ESO is immediately paid off if exercisable, or forfeited otherwise, which is negative exponentially distributed (i.e., $\mathbb{P}\{\tau_f \geq t\} = e^{-\lambda t}$ for $t \geq 0$) and independent of the filtration \mathcal{F} . For $t \in [0, T]$, let

$$\pi(t; T) = (S_t - K)^+ \mathbf{1}_{\{T_1 \leq t \leq T\}},$$

and define $\pi(\tau_H \wedge \tau_f; T)$ to be the ESO's payoff with independent stopping. The payoff $\pi(\tau_H \wedge \tau_f; T)$ for a fixed τ_f is the same as the no-exit payoff $\pi(\tau_H; T \wedge \tau_f)$. Under the condition $\{\tau_f = \tau\}$ for a given $\tau > 0$, we have

$$\mathbb{E} \left[e^{-r(\tau_H \wedge \tau_f)} \pi(\tau_H \wedge \tau_f; T) \mid \tau_f = \tau, \mathcal{F}_0 \right] = \mathbb{E} \left[e^{-r(\tau_H \wedge \tau)} \pi(\tau_H; T \wedge \tau) \mid \mathcal{F}_0 \right] = V(S; T \wedge \tau). \tag{3.4}$$

Hence, unconditioning (3.4), we obtain

$$\begin{aligned} V(S; T) &= \mathbb{E} \left[e^{-r(\tau_H \wedge \tau_f)} \pi(\tau_H; T \wedge \tau_f) \mid \mathcal{F}_0 \right] \\ &= \mathbb{E} \left[e^{-r\tau_H} \pi(\tau_H; T) \mathbf{1}_{\{\tau_f \geq T\}} \mid \mathcal{F}_0 \right] + \mathbb{E} \left[e^{-r(\tau_H \wedge \tau_f)} \pi(\tau_H; \tau_f) \mathbf{1}_{\{\tau_f < T\}} \mid \mathcal{F}_0 \right] \\ &= e^{-\lambda T} V^\circ(S; T) + \int_{T_1}^T \lambda e^{-\lambda t} V^\circ(S; t) dt, \end{aligned}$$

which proves the desired result. □

Remark 1. Unfortunately, the arbitrage price of such a contingent claim with payoff $\pi(\tau_H \wedge \tau_f; T)$ is *not* defined uniquely, because there exists no replicating \mathbb{P} -admissible trading strategy [24, p. 115]. This is due to the fact that $\tau_H \wedge \tau_f$ is *not* a stopping time of \mathcal{F} . The same situation also can be found in the valuation of *Canadian options*, which are contingent claims with the exponentially random maturity [7, 27]. Assuming there is no premium for the additional risk arising from independent stopping, Raupach [25] have showed that the contingent claim with payoff $\pi(\tau_H \wedge \tau_f; T)$ can be priced at its expected present value, just like a perfectly hedgeable contingent claim; see Appendix 5.4 of Raupach [25].

3.3. Valuation formula

To obtain an explicit expression of the conditional expectation in (3.2), let us introduce $\Phi_2(x, y; \gamma)$ for $(x, y) \in \mathbb{R}^2$, which denotes the bivariate standard normal cdf with the correlation coefficient γ ($|\gamma| < 1$), defined by

$$\Phi_2(x, y; \gamma) = \int_{-\infty}^x \int_{-\infty}^y \phi_2(u, v; \gamma) dv du$$

with

$$\phi_2(u, v; \gamma) = \frac{1}{2\pi\sqrt{1-\gamma^2}} \exp \left\{ -\frac{1}{2(1-\gamma^2)} (u^2 - 2\gamma uv + v^2) \right\}.$$

Then, we obtain

Theorem 1. *Let $V(S; T)$ be the value of ESO at grant date $t = 0$, with strike K , initial stock price $S_0 = S$, exit rate λ , vesting date T_1 and maturity T ($0 < T_1 < T$). Then, we have*

$$V(S; T) = e^{-\lambda T} V^\circ(S; T) + \int_{T_1}^T \lambda e^{-\lambda t} V^\circ(S; t) dt,$$

where $V^\circ(S; T)$ denotes the associated ESO value with no exit before maturity, which is given by

$$V^\circ(S; T) = Se^{-\delta T_1} \Phi(d_{11}) - Ke^{-rT_1} \Phi(d_{21}) + Se^{-\delta T} \psi_S - Ke^{-rT} \psi_K + (H - K) \psi_R. \quad (3.5)$$

The coefficients ψ_S , ψ_K and ψ_R are defined by

$$\begin{aligned} \psi_S &= \Phi_2(-d_{11}, d_{12}; -\rho) - \Phi_2(-d_{11}, d_{13}; -\rho) - \left(\frac{H}{S}\right)^{2(\alpha+1)} \{\Phi_2(d_{31}, d_{32}; \rho) - \Phi_2(d_{31}, d_{33}; \rho)\}, \\ \psi_K &= \Phi_2(-d_{21}, d_{22}; -\rho) - \Phi_2(-d_{21}, d_{23}; -\rho) - \left(\frac{H}{S}\right)^{2\alpha} \{\Phi_2(d_{41}, d_{42}; \rho) - \Phi_2(d_{41}, d_{43}; \rho)\}, \\ \psi_R &= \left(\frac{H}{S}\right)^{\alpha+\beta} \Phi_2(h_{11}, -h_{12}; -\rho) + \left(\frac{H}{S}\right)^{\alpha-\beta} \Phi_2(h_{21}, -h_{22}; -\rho), \end{aligned}$$

where

$$\begin{aligned} d_{11} &= d_+(S, H, T_1), & d_{12} &= d_+(S, K, T), & d_{13} &= d_+(S, H, T), \\ d_{21} &= d_-(S, H, T_1), & d_{22} &= d_-(S, K, T), & d_{23} &= d_-(S, H, T), \\ d_{31} &= d_+(H, S, T_1), & d_{32} &= d_+(H^2/K, S, T), & d_{33} &= d_+(H, S, T), \\ d_{41} &= d_-(H, S, T_1), & d_{42} &= d_-(H^2/K, S, T), & d_{43} &= d_-(H, S, T), \\ h_{11} &= h_+(H, S, T_1), & h_{12} &= h_+(H, S, T), & & \\ h_{21} &= h_-(H, S, T_1), & h_{22} &= h_-(H, S, T), & & \end{aligned}$$

and

$$\rho = \sqrt{\frac{T_1}{T}}.$$

Proof. By Lemma 2, it is sufficient to derive the expression for $V^\circ(S; T)$ in (3.5). The derivation is straightforward: From (3.2), we have

$$V^\circ(S; T) = e^{-rT_1} \left\{ \int_0^H c_H(S', T_1; T, K) p(S', T_1; S, 0) dS' + \int_H^\infty (S' - K) p(S', T_1; S, 0) dS' \right\}, \quad (3.6)$$

where for $S' > 0$,

$$p(S', T_1; S, 0) = \frac{1}{\sigma \sqrt{2\pi T_1} S'} \exp \left\{ -\frac{\left(\log(S'/S) - (r - \delta - \frac{1}{2}\sigma^2)T_1 \right)^2}{2\sigma^2 T_1} \right\} \quad (3.7)$$

is the lognormal probability density function (pdf) of $S_{T_1} \equiv S'$ starting from $S_0 = S$, which is often referred to as *Green's function* of the Black-Scholes PDE.

The second term in (3.6) is considered as a European call value with maturity T_1 and strike price K , but with a hurdle of level H ($> K$) for its exercise, which has been called a *gap option* in Rich [26]. Since $S' - K = (S' - H) + (H - K)$ for $S' \geq H$, this gap option can be further decomposed into a European vanilla call option with payoff $(S' - H)^+$ and a *cash-or-nothing* digital call option [24, p. 210] with payoff $(H - K) \mathbf{1}_{\{S' \geq H\}}$. The first two terms in the right-hand side of (3.5) can be derived from either this interpretation or direct

calculations. To derive the remaining terms in (3.5), we use the following relation among ϕ , Φ and Φ_2 :

$$\int_{-\infty}^d e^{cx} \Phi(a + bx) \phi(x) dx = e^{\frac{1}{2}c^2} \Phi_2 \left(d - c, \frac{a + bc}{\sqrt{1 + b^2}}; \frac{-b}{\sqrt{1 + b^2}} \right), \quad (3.8)$$

where $a, b, c, d \in \mathbb{R}$ are arbitrary constants; see, e.g., Kimura [19, p. 34] for the proof. Then, we have

$$\begin{aligned} & e^{-rT_1} \int_0^H c(S', T_1; T, K) p(S', T_1; S, 0) dS' \\ & \quad = Se^{-\delta T} \Phi_2(-d_{11}, d_{12}; -\rho) - Ke^{-rT} \Phi_2(-d_{21}, d_{22}; -\rho), \\ & e^{-rT_1} \int_0^H \left(\frac{H}{S'}\right)^{2\alpha} c\left(\frac{H^2}{S'}, T_1; T, K\right) p(S', T_1; S, 0) dS' \\ & \quad = Se^{-\delta T} \left(\frac{H}{S}\right)^{2(\alpha+1)} \Phi_2(d_{31}, d_{32}; \rho) - Ke^{-rT} \left(\frac{H}{S}\right)^{2\alpha} \Phi_2(d_{41}, d_{42}; \rho), \\ & e^{-rT_1} \int_0^H e^{-r(T-T_1)} \Phi(d_-(S', H, T - T_1)) p(S', T_1; S, 0) dS' = e^{-rT} \Phi_2(-d_{21}, d_{23}; -\rho), \\ & e^{-rT_1} \int_0^H e^{-r(T-T_1)} \left(\frac{H}{S'}\right)^{2\alpha} \Phi(d_-(H, S', T - T_1)) p(S', T_1; S, 0) dS' \\ & \quad = e^{-rT} \left(\frac{H}{S}\right)^{2\alpha} \Phi_2(d_{41}, d_{43}; \rho), \\ & e^{-rT_1} \int_0^H \left(\frac{H}{S'}\right)^{\alpha\pm\beta} \Phi(-h_{\pm}(H, S', T - T_1)) p(S', T_1; S, 0) dS' \\ & \quad = \left(\frac{H}{S}\right)^{\alpha\pm\beta} \Phi_2(h_{\pm}(H, S, T_1), -h_{\pm}(H, S, T); -\rho). \end{aligned}$$

Applying these equations into (3.2), we obtain the desired results after some simplifications. □

Remark 2. Clearly, the flat barrier H is *not* the optimal boundary of the problem (2.2), which means that the ESO value $V(S; T) \equiv V(S; T, H)$ in Theorem 1 always *underestimates* the true value for all $H (> K)$. Thus the possible maximum $\bar{V}(S; T) \equiv \max_H V(S; T, H)$ gives a lower bound of the exact ESO value; see Broadie and Detemple [5] for related bounds of an American call option value. In Section 6, we will use the maximum $\bar{V}(S; T)$ as an accuracy indicator of approximations for H .

Remark 3. To compute the integral in (3.3) numerically, we have to evaluate $V^\circ(S; t)$ for t close to T_1 , where the computation often fails if we directly use the expression in (3.5). The reason is that the bivariate normal cdf Φ_2 becomes highly correlated as $t \rightarrow T_1$, which causes degeneration. However, we can derive the analytical limit $V^\circ(S; T_1)$ directly from (3.5), which does not include Φ_2 : For $\rho = \pm 1$, it is easily verified that

$$\Phi_2(x, y; 1) = \Phi(x \wedge y) \quad \text{and} \quad \Phi_2(x, y; -1) = \begin{cases} \Phi(x) - \Phi(-y), & x + y > 0 \\ 0, & x + y \leq 0, \end{cases}$$

for $(x, y) \in \mathbb{R}^2$. Using these relations, we have

$$\begin{aligned}\lim_{T \rightarrow T_1} \psi_S &= \Phi(-d_{11}) - \Phi(-d_+(S, K, T_1)) \\ \lim_{T \rightarrow T_1} \psi_K &= \Phi(-d_{21}) - \Phi(-d_-(S, K, T_1)) \\ \lim_{T \rightarrow T_1} \psi_R &= 0,\end{aligned}$$

and hence

$$\lim_{T \rightarrow T_1} V^\circ(S; T) = Se^{-\delta T_1} \Phi(d_+(S, K, T_1)) - Ke^{-rT_1} \Phi(d_-(S, K, T_1)) = c(S, 0; T_1, K), \quad (3.9)$$

which is consistent with the expected result.

Theorem 2. Let $V_\infty(S)$ be the perpetual value of ESO with initial stock price S , exit rate λ and vesting date T_1 , i.e., $V_\infty(S) = \lim_{T \rightarrow \infty} V(S; T)$. Then, we have

$$V_\infty(S) = \begin{cases} \int_{T_1}^{\infty} \lambda e^{-\lambda t} V^\circ(S; t) dt, & \lambda > 0 \\ Se^{-\delta T_1} \Phi(d_+(S, \bar{S}, T_1)) - Ke^{-rT_1} \Phi(d_-(S, \bar{S}, T_1)) \\ \quad + \frac{\bar{S}}{\theta} \left(\frac{S}{\bar{S}} \right)^\theta \Phi(h_-(\bar{S}, S, T_1)), & \lambda = 0. \end{cases} \quad (3.10)$$

Proof. Letting $T \rightarrow \infty$ in (5.1), we have $\bar{S}_t = \bar{S}$ ($t \geq T_1$) for the perpetual case. Hence, from (5.2), $H = \bar{S}$, which is exact for the perpetual case. Similarly, letting $T \rightarrow \infty$ in (3.5), we have for $\lambda = 0$

$$\lim_{T \rightarrow \infty} V^\circ(S; T) = Se^{-\delta T_1} \Phi(d_{11}) - Ke^{-rT_1} \Phi(d_{21}) + (H - K) \left(\frac{H}{S} \right)^{\alpha - \beta} \Phi(h_{21}),$$

where we used the properties that $\lim_{y \rightarrow -\infty} \Phi_2(x, y; \gamma) = 0$ and $\lim_{y \rightarrow \infty} \Phi_2(x, y; \gamma) = \Phi(x)$. By virtue of the relations $\alpha - \beta = -\theta$, $H - K = \bar{S} - K = \bar{S}/\theta$, we obtain the desired result (3.10) for $\lambda = 0$. The result for $\lambda > 0$ directly comes from (3.3). \square

Remark 4. The perpetual result $V_\infty(S)$ in (3.10) is *exact* in the Black-Scholes-Merton formulation and it coincides with that independently obtained by using the quadratic approximation in the American framework; see Equations (33) and (34) in Kimura [17].

4. Exercise Time

In empirical studies of Carpenter [6] and Huddart and Lang [13], they displayed descriptive statistics for sample variables characterizing the exercise policy of ESOs, which include the average of the ratio of stock price at the time of ESO exercise to the strike price (i.e., M in this paper) and the average time of ESO exercise in years. The latter characteristic quantity still plays a critical role in the SFAS 123R proposal that recommends a modified Black-Scholes model where the stated maturity is replaced with the expected ESO life time. From samples of ten-year ESOs, the average exercise time has been reported as 5.83 years in Carpenter [6] and 3.4 years in Huddart and Lang [13].

Let X denote the exercise time of a vested ESO with strike K , initial stock price $S_0 = S$, vesting date T_1 and maturity T ($0 < T_1 < T$). Clearly, the exercise time of an unvested

ESO is defined as zero. Assume that $X = T$ when null payoff is paid at maturity. In addition, let $L(S; T) = \mathbb{E}[X | \mathcal{F}_0]$ denote the mean exercise time. To obtain an explicit expression for $L(S; T)$ in our barrier-option model, we begin with the analysis of a first passage time to the barrier: Let $\tau_H(S)$ denote the first passage time at which the stock price process S_t starting from $S_0 = S$ hits the barrier $H (> S)$ for the first time, i.e., $\tau_H(S) = \inf\{t \geq 0 | S_t \geq H, S_0 = S\}$. Then, by the theory of diffusion processes, we have

Lemma 3. Let $\bar{F}_H(t; S) = \mathbb{P}\{\tau_H(S) > t\}$ denote the complementary cdf of the first passage time $\tau_H(S)$, and define the Laplace transform of $\bar{F}_H(t; S)$ by

$$\bar{F}_H^*(\lambda; S) = \int_0^\infty e^{-\lambda t} \bar{F}_H(t; S) dt,$$

for $\lambda \in \mathbb{C}$ ($\text{Re}(\lambda) > 0$). Then, we have

$$\bar{F}_H(t; S) = \Phi\left(\frac{\log(H/S) - \alpha\sigma^2 t}{\sigma\sqrt{t}}\right) - \left(\frac{H}{S}\right)^{2\alpha} \Phi\left(-\frac{\log(H/S) + \alpha\sigma^2 t}{\sigma\sqrt{t}}\right), \quad (4.1)$$

and

$$\bar{F}_H^*(\lambda; S) = \frac{1}{\lambda} \left\{ 1 - \left(\frac{H}{S}\right)^{\alpha - \beta^*} \right\}, \quad (4.2)$$

where the parameters α and β^* are given by

$$\alpha = \frac{1}{\sigma^2} \left(r - \delta - \frac{1}{2}\sigma^2 \right), \quad \beta^* = \sqrt{\alpha^2 + \frac{2\lambda}{\sigma^2}}. \quad (4.3)$$

Proof. Under the risk-neutral probability measure \mathbb{P} , consider an arithmetic Brownian motion process $(W_t^\mu)_{t \geq 0}$ with drift μ , volatility σ and $W_0^\mu = 0$ (a.s.), i.e., $dW_t^\mu = \mu dt + \sigma dW_t$ for $t \geq 0$. Let $B (> 0)$ denote an upper barrier, and define $\tilde{\tau}_B$ to be the first passage time of the process W_t^μ to the barrier B , i.e., $\tilde{\tau}_B = \inf\{t \geq 0 | W_t^\mu \geq B, W_0^\mu = 0\}$. By virtue of the reflection principle, it has been proved (see, e.g., Cox and Miller [9, p. 221]) that

$$\mathbb{P}\{\tilde{\tau}_B > t\} = \Phi\left(\frac{B - \mu t}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2\mu B}{\sigma^2}\right) \Phi\left(-\frac{B + \mu t}{\sigma\sqrt{t}}\right). \quad (4.4)$$

From the fact that the underlying stock price process is formulated as the geometric Brownian motion $S_t = Se^{W_t^\mu}$ with $S_0 = S$ under the measure \mathbb{P} , we have

$$\mathbb{P}\{\tau_H(S) > t\} = \mathbb{P}\{\tilde{\tau}_{\log(H/S)} > t\}.$$

Hence, we obtain (4.1) by substituting $B = \log(H/S)$ and $\mu = r - \delta - \frac{1}{2}\sigma^2 = \alpha\sigma^2$ into (4.4).

Let $f_H^*(\lambda; S)$ and $\tilde{f}_B^*(\lambda)$ denote the Laplace transforms of the pdfs of $\tau_H(S)$ and $\tilde{\tau}_B$, respectively. Then, it also has been known (Equation (75) in Cox and Miller [9, p. 221]) that

$$\tilde{f}_B^*(\lambda) = \exp\left\{ \frac{B}{\sigma^2} \left(\mu - \sqrt{\mu^2 + 2\lambda\sigma^2} \right) \right\}.$$

Using the equivalence of $\tau_H(S)$ and $\tilde{\tau}_{\log(H/S)}$, we have

$$f_H^*(\lambda; S) = \tilde{f}_{\log(H/S)}^*(\lambda) = \left(\frac{H}{S}\right)^{\alpha - \beta^*}.$$

The elementary property $\bar{F}_H^*(\lambda; S) = (1 - f_H^*(\lambda; S))/\lambda$ yields (4.2). □

Consider the case that no exit occurs before maturity, in which a vested ESO can be exercised either at (i) the vesting date T_1 if $S_{T_1} \geq H$, or (ii) $\min\{T_1 + \tau_H(S_{T_1}), T\}$ otherwise. Hence, the exercise time X can be defined in terms of τ_H as

$$\begin{aligned} X &= T_1 \mathbf{1}_{\{S_{T_1} \geq H\}} + \min\{T_1 + \tau_H(S_{T_1}), T\} \mathbf{1}_{\{S_{T_1} < H\}} \\ &= T_1 \mathbf{1}_{\{S_{T_1} \geq H\}} + \left(\min\{\tau_H(S_{T_1}), T - T_1\} + T_1 \right) \mathbf{1}_{\{S_{T_1} < H\}} \\ &= T_1 + \min\{\tau_H(S_{T_1}), T - T_1\} \mathbf{1}_{\{S_{T_1} < H\}}. \end{aligned} \quad (4.5)$$

Theorem 3. Let $\bar{G}(t) = \mathbb{P}\{X > t\}$ denote the complementary cdf of X . Then, we have

$$\bar{G}(t) = \Phi_2 \left(-d_{21}, -d_-(S, H, t); \sqrt{\frac{T_1}{t}} \right) - \left(\frac{H}{S} \right)^{2\alpha} \Phi_2 \left(d_{41}, -d_-(H, S, t); -\sqrt{\frac{T_1}{t}} \right).$$

Proof. From (4.5), we have

$$\bar{G}(t) = \mathbb{P}\{\min\{\tau_H(S_{T_1}), T - T_1\} \mathbf{1}_{\{S_{T_1} < H\}} > t - T_1\}, \quad t \geq 0,$$

which yields that $\bar{G}(t) = 1$ for $t < T_1$, and $\bar{G}(t) = 0$ for $t \geq T$. By the equivalence relation for $t \in [T_1, T)$

$$\left\{ \min\{\tau_H(S_{T_1}), T - T_1\} \mathbf{1}_{\{S_{T_1} < H\}} > t - T_1 \right\} \equiv \left\{ \tau_H(S_{T_1}) \mathbf{1}_{\{S_{T_1} < H\}} > t - T_1 \right\},$$

we obtain for $t \in [T_1, T)$

$$\begin{aligned} \bar{G}(t) &= \mathbb{P}\{\tau_H(S_{T_1}) \mathbf{1}_{\{S_{T_1} < H\}} > t - T_1\} \\ &= \int_0^H \bar{F}_H(t - T_1; S') p(S', T_1; S, 0) dS' \\ &= \Phi_2 \left(-d_{21}, -d_-(S, H, t); \sqrt{\frac{T_1}{t}} \right) - \left(\frac{H}{S} \right)^{2\alpha} \Phi_2 \left(d_{41}, -d_-(H, S, t); -\sqrt{\frac{T_1}{t}} \right), \end{aligned}$$

from Lemma 3 and (3.8). □

Taking the limit $t \rightarrow T_1$ in $\bar{G}(t)$ and using the relations

$$\Phi_2(x, x; 1) = \Phi(x) \quad \text{and} \quad \Phi_2(x, -x; -1) = 0$$

for $x \in \mathbb{R}$, we have

$$\lim_{t \rightarrow T_1} \bar{G}(t) = \Phi(-d_{21}) = \mathbb{P}\{S_{T_1} < H\}. \quad (4.6)$$

Also, letting $t \rightarrow T$, we have

$$\lim_{t \rightarrow T} \bar{G}(t) = \Phi_2(-d_{21}, -d_{23}; \rho) - \left(\frac{H}{S} \right)^{2\alpha} \Phi_2(d_{41}, -d_{43}; -\rho). \quad (4.7)$$

The explicit expression for the complementary cdf \bar{G} in Theorem 3 leads to

Theorem 4. Let $L(S; T)$ denote the mean exercise time of a vested ESO with strike K , initial stock price $S_0 = S$, exit rate λ , vesting date T_1 and maturity T ($0 < T_1 < T$). Then, we have

$$L(S; T) = T_1 e^{-\lambda T_1} + \int_{T_1}^T e^{-\lambda t} \left\{ \Phi_2 \left(-d_{21}, -d_-(S, H, t); \sqrt{\frac{T_1}{t}} \right) - \left(\frac{H}{S} \right)^{2\alpha} \Phi_2 \left(d_{41}, -d_-(H, S, t); -\sqrt{\frac{T_1}{t}} \right) \right\} dt.$$

Proof. First assume no exit before maturity, for which denote the mean exercise time by $L^\circ(S; T)$. Then, we have

$$L^\circ(S; T) = \int_0^\infty \bar{G}(t) dt = T_1 + \int_{T_1}^T \bar{G}(t) dt. \tag{4.8}$$

Note that $\bar{G}(t)$ does *not* contain the variable T in itself. By the argument for independent exit similar to Theorem 1, we have the integral expression for $L(S; T)$ as

$$L(S; T) = e^{-\lambda T} L^\circ(S; T) + \int_{T_1}^T \lambda e^{-\lambda u} L^\circ(S; u) du. \tag{4.9}$$

Then, combining (4.8) and (4.9), we obtain

$$\begin{aligned} L(S; T) &= e^{-\lambda T} \left\{ T_1 + \int_{T_1}^T \bar{G}(t) dt \right\} + \int_{T_1}^T \lambda e^{-\lambda u} \left\{ T_1 + \int_{T_1}^u \bar{G}(t) dt \right\} du \\ &= e^{-\lambda T} \left\{ T_1 + \int_{T_1}^T \bar{G}(t) dt \right\} + \lambda T_1 \int_{T_1}^T e^{-\lambda u} du + \lambda \int_{T_1}^T \bar{G}(t) \int_t^T e^{-\lambda u} du dt \\ &= e^{-\lambda T} \left\{ T_1 + \int_{T_1}^T \bar{G}(t) dt \right\} + T_1 (e^{-\lambda T_1} - e^{-\lambda T}) + \int_{T_1}^T \bar{G}(t) (e^{-\lambda t} - e^{-\lambda T}) dt \\ &= T_1 e^{-\lambda T_1} + \int_{T_1}^T e^{-\lambda t} \bar{G}(t) dt, \end{aligned}$$

which completes the proof. □

Corollary 1. The mean exercise time $L(S; T)$ is

- (i) a concave increasing function of maturity T ; and
- (ii) a convex decreasing function of exit rate λ .

Proof. From (4.9), we see that

$$\begin{cases} \frac{\partial L}{\partial T} = e^{-\lambda T} \bar{G}(T) > 0 \\ \frac{\partial^2 L}{\partial T^2} = -\lambda e^{-\lambda T} \bar{G}(T) + e^{-\lambda T} \frac{d\bar{G}}{dT} < 0, \end{cases}$$

and

$$\begin{cases} \frac{\partial L}{\partial \lambda} = - \left(T_1^2 e^{-\lambda T_1} + \int_{T_1}^T t e^{-\lambda t} \bar{G}(t) dt \right) < 0 \\ \frac{\partial^2 L}{\partial \lambda^2} = T_1^3 e^{-\lambda T_1} + \int_{T_1}^T t^2 e^{-\lambda t} \bar{G}(t) dt > 0, \end{cases}$$

which prove the assertions (i) and (ii), respectively. □

Using Lemma 3, we can derive a closed-form solution of the mean exercise time for the perpetual ESO, which gives an *upper bound* of $L(S; T)$ for the associated finite-lived ESO, due to the property (i) in Corollary 1. Similarly to the perpetual value $V_\infty(S)$, it also gives the exact result for the perpetual case in the Black-Scholes-Merton formulation.

Theorem 5. *Let $L_\infty(S)$ denote the mean exercise time of the perpetual ESO with strike K , initial stock price S , exit rate λ and vesting date T_1 , i.e., $L_\infty(S) = \lim_{T \rightarrow \infty} L(S; T)$. Then, for $\lambda > 0$ we have*

$$L_\infty(S) = T_1 e^{-\lambda T_1} + \frac{1}{\lambda} \left\{ \Phi(-d_-(S, \bar{S}, T_1)) - \left(\frac{S}{\bar{S}}\right)^{\alpha - \beta^*} \Phi(h_-^*(\bar{S}, S, T_1)) \right\}, \quad (4.10)$$

where

$$h_-^*(\bar{S}, S, T_1) = \frac{\log(\bar{S}/S) - \beta^* \sigma^2 T_1}{\sigma \sqrt{T_1}},$$

and β^* is defined in (4.3). Also, let $L_\infty^\circ(S) = \lim_{T \rightarrow \infty} L^\circ(S; T)$ for $\lambda = 0$. Then, we have

$$L_\infty^\circ(S) = \begin{cases} T_1 - \log\left(\frac{\bar{S}}{S}\right) \left\{ T_1 \Phi(-d_-(S, \bar{S}, T_1)) - \frac{\sqrt{T_1}}{\alpha \sigma} \phi(-d_-(S, \bar{S}, T_1)) \right\}, & \alpha > 0 \\ +\infty, & \alpha \leq 0. \end{cases}$$

Proof. First note that the barrier level is given by $H = \bar{S}$ for the perpetual case, as shown in Theorem 2. From the relation

$$L(S; T) = T_1 e^{-\lambda T_1} + \int_{T_1}^T e^{-\lambda t} \bar{G}(t) dt,$$

we have

$$\begin{aligned} L_\infty(S) &= T_1 e^{-\lambda T_1} + \int_{T_1}^{\infty} e^{-\lambda t} \mathbb{P} \left\{ \tau_H(S_{T_1}) \mathbf{1}_{\{S_{T_1} < H\}} > t - T_1 \right\} dt \\ &= T_1 e^{-\lambda T_1} + e^{-\lambda T_1} \int_0^{\infty} e^{-\lambda t} \mathbb{P} \left\{ \tau_H(S_{T_1}) \mathbf{1}_{\{S_{T_1} < H\}} > t \right\} dt \\ &= T_1 e^{-\lambda T_1} + e^{-\lambda T_1} \int_0^{\infty} e^{-\lambda t} \int_0^H \bar{F}_H(t; S') p(S', T_1; S, 0) dS' dt \\ &= T_1 e^{-\lambda T_1} + e^{-\lambda T_1} \int_0^H \bar{F}_H^*(\lambda; S') p(S', T_1; S, 0) dS' \\ &= T_1 e^{-\lambda T_1} + \frac{1}{\lambda} \left\{ \Phi(-d_-(S, \bar{S}, T_1)) - \left(\frac{S}{\bar{S}}\right)^{\alpha - \beta^*} \Phi(h_-^*(\bar{S}, S, T_1)) \right\}, \end{aligned}$$

which proves (4.10). We can obtain $L_\infty^\circ(S)$ for $\lambda = 0$ either by letting $\lambda \rightarrow 0$ in (4.10) with the aid of l'Hospital's rule, or by applying the well-known result [9, Equation (76) in p. 222]

$$\mathbb{E}[\tau_H(S')] = \int_0^{\infty} \bar{F}_H(t; S') dt = \begin{cases} B/\mu, & \mu > 0 \\ +\infty, & \mu \leq 0 \end{cases}$$

into

$$L_\infty^\circ(S) = T_1 + \int_0^H \mathbb{E}[\tau_H(S')] p(S', T_1; S, 0) dS',$$

where $B = \log(H/S')$ and $\mu = \alpha \sigma^2$. □

5. Heuristics for the Barrier Level

In all fairness, arbitrary parameters should be excluded from the valuation model if at all possible. To determine the early exercise level H in a rational manner, we first give a simple approximation for the early exercise boundary $(\bar{S}_t)_{t \in [T_1, T]}$. There have been several approximations developed for the early exercise boundary, most of which are based on its short-time asymptotics near maturity and hence they are suitable for short-term options; see Barone-Adesi and Whaley [2], Bjerksund and Stensland [4] and Goodman and Ostrov [12]. Also see Kimura [20] for a comprehensive review. However, for the purpose of obtaining a practical approximation for the level H , we need a much simpler approximation for the boundary $(\bar{S}_t)_{t \in [T_1, T]}$ than those previously established approximations.

Based on the observations in Figure 1, we introduce two naive assumptions on the shape of $(\bar{S}_t)_{t \in [T_1, T]}$ such that (i) the boundary is a square-root function of the remaining time to maturity $\tau = T - t$, starting from \bar{S}_T given in (2.4) at $\tau = 0$, and (ii) the boundary already reaches at the perpetual value \bar{S} given in (2.6) at $\tau = T - T_1 < \infty$, i.e., $\bar{S}_{T_1} = \bar{S}$. The assumption (i) reflects the boundary behavior close to maturity, while the assumption (ii) is due to the long period up to maturity for the ESO. From these two assumptions, we propose a *square-root approximation* for the early exercise boundary as

$$\bar{S}_t = \bar{S}_T + (\bar{S} - \bar{S}_T) \sqrt{\frac{T - t}{T - T_1}}, \quad t \in [T_1, T]. \tag{5.1}$$

No doubt, complete accuracy cannot be expected from this rough approximation. In particular, for $\bar{S} \gg \bar{S}_T$, it overestimates (underestimates) the true value for $t \rightarrow T_1$ ($t \rightarrow T$); see Figure 2. The principal purpose in proposing the square-root approximation in (5.1) is, however, to generate a concise approximation for H , or equivalently for M . It is a matter of course that there is no definitive method for determining the *flat* boundary H that approximates the *curved* boundary $(\bar{S}_t)_{t \in [T_1, T]}$. Among a few alternatives, we adopt here a natural approximation such that H is given by the *average height* of the curve, i.e.,

$$H = \frac{1}{T - T_1} \int_{T_1}^T \bar{S}_t dt = \frac{1}{3} \bar{S}_T + \frac{2}{3} \bar{S}. \tag{5.2}$$

Clearly, H is equal to the level dividing the difference between \bar{S}_T and \bar{S} internally in the ratio 2:1, and hence $\bar{S} > H > \bar{S}_T \geq K$. Figure 2 shows the normalized early exercise boundary (bold line) and its square-root approximation (gray line) for a particular case

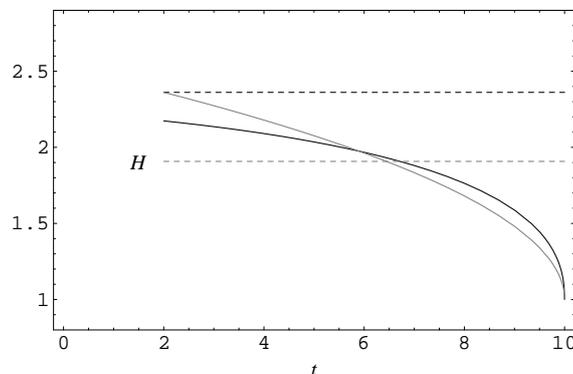


Figure 2: A square-root approximation for the normalized early exercise boundary $(\bar{S}_t/K)_{t \in [T_1, T]}$ ($T_1 = 2$, $T = 10$, $r = 0.04$, $\delta = 0.05$, $\sigma = 0.3$)

with $r = 0.04$, $\delta = 0.05$ and $\sigma = 0.3$. Note that $\bar{S}_T = K$ for this case. The dashed line represents the perpetual value \bar{S} just as in Figure 1, and the dashed gray line represents the early exercise level H .

The simple early exercise policy with H would be useful in practical business, and above all things we will see in Section 6 that our barrier-option model with H in (5.2) exactly follows benchmark results generated by an American binomial-tree model for ESOs.

Remark 5. Figure 3 illustrates a 3-dimensional surface of the early exercise multiple

$$M = \frac{H}{K} = \frac{1}{3} \max\left(1, \frac{r}{\delta}\right) + \frac{2}{3} \frac{\theta}{\theta - 1} \quad (5.3)$$

on the r - δ plane for $\sigma = 0.3$ and $(r, \delta) \in (0, 0.1] \times (0, 0.1]$. We have actually depicted $\min(M, 6)$ in the figure, in which we see that the slope of the surface tends to be rapidly steep as $\delta \rightarrow 0$ because $\lim_{\delta \rightarrow 0} M = \infty$. This effect of δ on M is consistent with an empirical observation in Bettis et al. [3] such that ESOs are exercised earlier in firms with higher dividend yields. From Figure 3, we see that M is sensitive to r if δ is small, and that M is relatively insensitive to the parameters r and δ if $r < \delta$, being flat on that region; cf. Figure 1. The insensitivity to (r, δ) is a desirable property of the early exercise policy in a versatile economy.

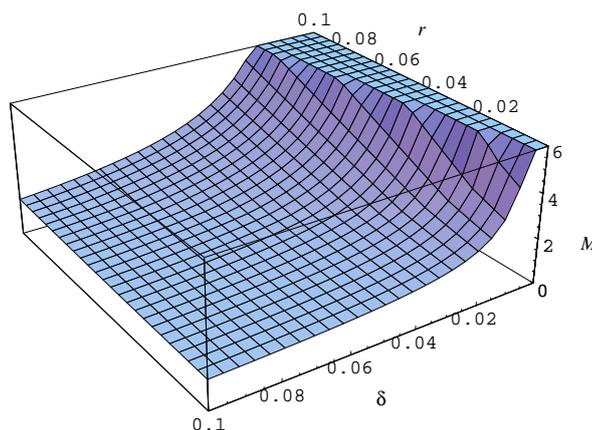


Figure 3: 3D-surface of the early exercise multiple M on the r - δ plane ($\sigma = 0.3$, $(r, \delta) \in (0, 0.1] \times (0, 0.1]$)

6. Computational Results

First we examine the accuracy of the formula for the ESO value $V(S; T)$ in Theorem 1. To see the quality of the formula, we numerically compare it with benchmark results generated by a binomial-tree model, which extends the standard American option model by introducing delayed vesting as well as random exit. The extended American binomial-tree model can be easily implemented by modifying the backward induction algorithm for valuing the standard American option; see the appendices of [1, 6, 27] for specific algorithms.

To accelerate the convergence of binomial-tree values, we use the 3-point Richardson scheme of extrapolating the 1000-, 2000- and 3000-period binomial values. As a standard set of parameters, we use $S = K = 1$, $T_1 = 2$ and $T = 10$ unless otherwise stated. We saw in Figure 1 that the early exercise boundary is relatively less sensitive to risk-free rate r than the other market parameters, and hence we fix $r = 0.03$ as a sample value. Also, in

order to accelerate the numerical integration appeared in $V(S; T)$, we approximate $V^\circ(S; u)$ for $u \in [T_1, T]$ by a cubic spline curve interpolating $V^\circ(S; T_1) = c(S, 0; T_1, K)$ at $u = T_1$ (see Remark 3) and values $V^\circ(S; u)$ at adjacent points $u = T_1 + 1, \dots, T$. The spline approximation quickly generates a very accurate result, of which first four digits coincide with those of the direct numerical integration result.

Table 1: A comparison of values of executive stock options ($T = 10, T_1 = 2, S = K = 1, r = 0.03, \lambda = 0.0$)

δ	σ	binomial	$\bar{V}^\circ(S; T)$ (%)	$V^\circ(S; T)$ (%)	quadratic (%)
0.02	0.2	0.2429	0.2424 (-0.21)	0.2415 (-0.58)	0.2474 (1.85)
	0.3	0.3406	0.3397 (-0.26)	0.3385 (-0.62)	0.3466 (1.76)
	0.4	0.4323	0.4310 (-0.30)	0.4298 (-0.58)	0.4395 (1.67)
0.03	0.2	0.2043	0.2034 (-0.44)	0.2034 (-0.44)	0.2084 (2.01)
	0.3	0.3010	0.2997 (-0.43)	0.2996 (-0.47)	0.3069 (1.96)
	0.4	0.3915	0.3900 (-0.38)	0.3899 (-0.41)	0.3991 (1.94)
0.04	0.2	0.1743	0.1734 (-0.52)	0.1732 (-0.63)	0.1776 (1.89)
	0.3	0.2682	0.2671 (-0.41)	0.2670 (-0.45)	0.2737 (2.05)
	0.4	0.3569	0.3554 (-0.42)	0.3554 (-0.42)	0.3641 (2.02)
0.05	0.2	0.1499	0.1492 (-0.47)	0.1484 (-1.00)	0.1524 (1.67)
	0.3	0.2406	0.2395 (-0.46)	0.2389 (-0.71)	0.2451 (1.87)
	0.4	0.3268	0.3256 (-0.37)	0.3251 (-0.52)	0.3333 (1.99)

Table 2: A comparison of values of executive stock options ($T = 10, T_1 = 2, S = K = 1, r = 0.03, \lambda = 0.1$)

δ	σ	binomial	$\bar{V}(S; T)$ (%)	$V(S; T)$ (%)	quadratic (%)
0.02	0.2	0.1717	0.1715 (-0.12)	0.1709 (-0.47)	0.1741 (1.40)
	0.3	0.2425	0.2422 (-0.12)	0.2413 (-0.49)	0.2458 (1.36)
	0.4	0.3097	0.3093 (-0.13)	0.3083 (-0.45)	0.3139 (1.36)
0.03	0.2	0.1466	0.1463 (-0.20)	0.1462 (-0.27)	0.1492 (1.77)
	0.3	0.2167	0.2163 (-0.18)	0.2161 (-0.28)	0.2204 (1.71)
	0.4	0.2831	0.2827 (-0.14)	0.2824 (-0.25)	0.2880 (1.73)
0.04	0.2	0.1268	0.1264 (-0.32)	0.1264 (-0.32)	0.1291 (1.81)
	0.3	0.1950	0.1948 (-0.10)	0.1947 (-0.15)	0.1988 (1.95)
	0.4	0.2604	0.2599 (-0.19)	0.2599 (-0.19)	0.2652 (1.84)
0.05	0.2	0.1103	0.1101 (-0.18)	0.1100 (-0.27)	0.1123 (1.81)
	0.3	0.1767	0.1763 (-0.23)	0.1762 (-0.28)	0.1799 (1.81)
	0.4	0.2403	0.2400 (-0.12)	0.2400 (-0.12)	0.2449 (1.91)

Tables 1 and 2 compare our approximation for $V(S; T)$ with the benchmark result (referred to as “binomial”), the accuracy indicator $\bar{V}(S; T) = \max_H V(S; T, H)$ shown in Remark 2 and the quadratic approximation of Kimura [17, Theorems 1 and 2] (referred to as “quadratic”) for some combinations of the parameters δ, σ and λ , where numerical values in parentheses are relative percentage errors with respect to the benchmark. No doubt, the accuracy indicator $\bar{V}(S; T)$ gives a theoretical upper limit of all over the barrier-option

model, which is calculated from $V(S; T)$ (or $V^\circ(S; T)$ for $\lambda = 0$) in Theorem 1 by applying the Nelder-Mead method. The quadratic approximation for $V^\circ(S; T)$ is given in [17, Equation (19)] and $V(S; T)$ has the same integral expression as in Lemma 2.

These tables show that

- (i) our approximation is stably accurate, regardless of the parameters δ and σ ;
- (ii) the heuristic approximation (5.2) for H works well, because $V(S; T)$ is very close behind the theoretical upper limit $\bar{V}(S; T)$; and
- (iii) the quality of our approximation is much better than the quadratic approximation.

The second property (iii) is a little bit unexpected result, since the barrier option models have been rather considered as an *ad hoc* approximation in the previous literature. From the tables, we see that our approximations always underestimate the benchmark results as indicated in Remark 2 and the absolute values of the relative percentage errors are less than 1% for all cases, whereas the quadratic approximations overestimate the benchmarks and the relative percentage errors are less than about 2%.

To see the effects of the length of maturity to the ESO value, Figure 4(a) illustrates the values $V(S; T)$ for $T = 5, 10, 15$ as functions of exit rate λ , where we added the perpetual value $V_\infty(S)$ for $T = \infty$ drawn in a dashed line. We observe from this figure that the perpetual value is an upper bound of $V(S; T)$ for $T < \infty$, which is almost insensitive to the maturity date, e.g., if $\lambda > 0.3$ for $T = 10$. This robustness is due to the fact that the ESO exercise for large λ is actually driven by exit not by maturity. Hence, for large λ , $V(S; T)$ can be well approximated by $V_\infty(S)$ even for finite-lived cases, which leads to the fourth properties:

- (iv) the larger exit rate λ , the better our approximation.

Figure 4(b) also illustrates the effects of maturity to the mean exercise time $L(S; T)$, where the dashed line represents $L_\infty(S)$ for $T = \infty$. Clearly, this figure provides numerical validation of Corollary 1. In the same way as for the ESO value, we apply the cubic spline approximation to the numerical integration appeared in $L(S; T)$. That is, we approximate the complementary cdf $\bar{G}(t)$ for $t \in [T_1, T]$ by a cubic spline curve evaluated at a sequence of discrete points $t = T_1, T_1 + \Delta t, \dots, T$. We used the time step $\Delta t = 0.1$ in the experiment. From Figure 4(b), we see that the mean exercise time $L(S; T)$ also has the robustness similar to $V(S; T)$ for large λ , and further that the mean exercise time is differently affected by the length of maturity especially for $\lambda \approx 0$. We have $L^\circ(S; 5) = 4.8073$ for the short-term case

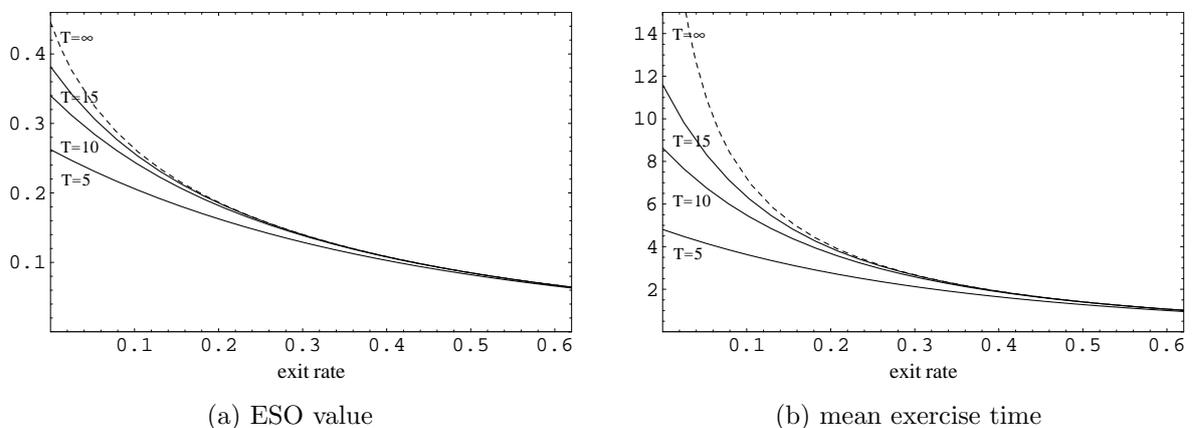


Figure 4: Values and mean exercise times of executive stock options with different maturities ($T_1 = 2$, $S = K = 1$, $r = 0.05$, $\delta = 0.03$, $\sigma = 0.3$)

($T = 5$) and $L^\circ(S; 10) = 8.6316$ for the standard-term case ($T = 10$), which implies that the ESO exercise occurs mostly at maturity for $T = 5$, but for $T = 10$ it may occur before maturity by breaching the barrier. We also have $L_\infty^\circ = \infty$ because $\alpha < 0$ for this particular case.

7. Concluding Remarks

If an ESO holder acts in accordance with the results in this paper, (s)he exercises the vested ESO early before maturity either when the stock price first hits the boundary H , or when (s)he leaves the firm with exit rate λ . The second factor is assumed to be subsidiary to the first one in this paper, so that the approximate formula for H in (5.2) does *not* contain the parameter λ in it. This is plausible if the employment contract is terminated involuntarily, because the ESO holder cannot know in advance the information on exit that occurred independently of the stock price process. However, for a voluntary exit, (s)he has in mind some information, and hence the boundary level H explicitly depends on the voluntary exit rate; see Sircar and Xiong [27] and Leung and Sircar [22]. Actually, the dependence on λ in H does not matter so much to the ESO value $V(S; T)$, since the exit effect is accurately taken into the value through the integral relation (3.3). However, it matters a great deal to the exercise policy of the ESO holder. From the view point of ESO holders, it is important to develop a detailed model accounting for the causes of employment termination that separate voluntary and involuntary exits. As Leung [21] has pointed out, such a model generally assumes sufficient and well-segmented empirical data, which is a principal reason why previous literature has adopted reduced-form modeling. It would be a direction of future research to develop a valuation model for ESOs with a dual-structured employment termination consisting of voluntary and involuntary exits, if more complete data is available.

From Tables 1 and 2, we saw that our approximations always underestimate the benchmark results as discussed in Remark 2, while the quadratic approximations in Kimura [17] always overestimate them. In the previous research on the quadratic approximation (QA) with applications to option pricing, it has been reported that QA has a general tendency to overestimate the benchmark results, though any convincing reasons are not given yet.

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References

- [1] M. Ammann and R. Seiz: Valuing employee stock options: Does the model matter? *Financial Analysts Journal*, **60** (2004), 21–37. doi:10.2139/ssrn.499782.
- [2] G. Barone-Adesi and R.E. Whaley: Efficient analytic approximation of American option values. *Journal of Finance*, **42** (1987), 301–320. doi:10.1111/j.1540-6261.1987.tb02569.x.
- [3] J.C. Bettis, J.M. Bizjak, and M.L. Lemmon: Exercise behavior, valuation, and the

- incentive effects of employee stock options. *Journal of Financial Economics*, **76** (2005), 445–470. doi:10.1016/j.jfineco.2004.05.006.
- [4] P. Bjerksund and G. Stensland: Closed-form approximation of American options. *Scandinavian Journal of Management*, **9** (1993), S87–S99. doi:10.1016/0956-5221(93)90009-H.
- [5] M. Broadie and J. Detemple: American option valuation: new bounds, approximations, and a comparison of existing methods. *Review of Financial Studies*, **9** (1996), 1211–1250. doi:10.1093/rfs/9.4.1211.
- [6] J.N. Carpenter: The exercise and valuation of executive stock options. *Journal of Financial Economics*, **48** (1998), 127–158. doi:10.1016/S0304-405X(98)00006-3.
- [7] P. Carr: Randomization and the American put. *Review of Financial Studies*, **11** (1998), 597–626. doi:10.1093/rfs/11.3.597.
- [8] P. Carr and V. Linetsky: The valuation of executive stock options in an intensity-based framework. *European Finance Review*, **4** (2000), 211–230. doi:10.1023/A:1011441824560.
- [9] D.R. Cox and H.D. Miller: *The Theory of Stochastic Processes* (Chapman & Hall, London, 1965).
- [10] J.C. Cox, S. Ross, and M. Rubinstein: Option pricing: a simplified approach. *Journal of Financial Economics*, **7** (1979), 229–263. doi:10.1016/0304-405X(79)90015-1.
- [11] J. Cvitanić, Z. Wiener, and F. Zapatero: Analytic pricing of employee stock options. *Review of Financial Studies*, **21** (2008), 683–724. doi:10.1093/rfs/hhm065.
- [12] J. Goodman and D.N. Ostrov: On the early exercise boundary of the American put option. *SIAM Journal on Applied Mathematics*, **62** (2002), 1823–1835. doi:10.1137/S0036139900378293.
- [13] S. Huddart and M. Lang: Employee stock option exercises: an empirical analysis. *Journal of Accounting and Economics*, **20** (1996), 5–43. doi:10.1016/0165-4101(95)00409-2.
- [14] J.C. Hull and A. White: How to value employee stock options. *Financial Analysts Journal*, **60** (2004), 114–119. doi:10.2469/faj.v60.n1.2596.
- [15] L.P. Jennergren and B. Näslund: A comment on “Valuation of executive stock options and the FASB proposal”. *The Accounting Review*, **68** (1993), 179–183.
- [16] I.J. Kim: The analytic valuation of American puts. *Review of Financial Studies*, **3** (1990), 547–572. doi:10.1093/rfs/3.4.547.
- [17] T. Kimura: A quadratic approximation for valuing executive stock options. *European Journal of Operational Research*, **207** (2010), 1368–1379. doi:10.1016/j.ejor.2010.06.041.
- [18] T. Kimura: Alternative randomization for valuing American options. *Asia-Pacific Journal of Operational Research*, **27** (2010), 167–187. doi:10.1142/S0217595910002624.
- [19] T. Kimura: *Financial Mathematics* (Minerva Shobo, Kyoto, 2011) (in Japanese).
- [20] T. Kimura: Approximations for the early exercise boundary of American options: A review. *Transactions of the Japan Society for Industrial and Applied Mathematics*, **24** (2014), 275–292 (in Japanese). doi:10.11540/jsiamt.24.3_275.
- [21] T. Leung: Employee stock options: accounting for optimal hedging, suboptimal exercises, and contractual restrictions (Working paper, Dept. of Applied Mathematics & Statistics, Johns Hopkins University, 2008).

- [22] T. Leung and R. Sircar: Accounting for risk aversion, vesting, job termination risk and multiple exercises in valuation of employee stock options. *Mathematical Finance*, **19** (2009), 99–128. doi:10.1111/j.1467-9965.2008.00359.x.
- [23] H.P. McKean: Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics. *Industrial Management Review*, **6** (1965), 32–39.
- [24] M. Musiela and M. Rutkowski: *Martingale Methods in Financial Modelling* (Springer, New York, 1997).
- [25] P. Raupach: The valuation of employee stock options – How good is the standard? (Working Paper, Goethe University Frankfurt am Main, 2003).
- [26] D.R. Rich: The mathematical foundations of barrier option-pricing theory. *Advances in Futures and Options Research*, **7** (1994), 267–311.
- [27] R. Sircar and W. Xiong: A general framework for evaluating executive stock options. *Journal of Economic Dynamics & Control*, **31** (2007), 2317–2349. doi:10.1016/j.jedc.2006.07.004.

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