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A Valid Perturbation and Lexicographic Ordering in Optimization over the Efficient Set

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1 Introduction

Let X be a set of feasible solutions in $\mathbb{R}^n$ defined by a system of linear inequalities such that $0 \in \mathrm{int}(X)$. Also that $c^1, c^2, \cdots, c^k$ be k independent vectors in $\mathbb{R}^n$, and let X° be the efficient set of the following multi-objective program

$$\left|\begin{array}{ll} \text{maximize} & \langle c^i, x\rangle,\ i = 1, \cdots, k \\ \text{subject to} & x \in X. \end{array}\right.$$

We are interested in solving the following program

$$\text{minimize}\{\langle c, x\rangle \ | x \in X^\circ\}. \tag{1}$$

Define

$$d = \sum_{i=1}^{k} c^i,\ c^i(s) = c^i + sd,$$
$$C^s = \{x \in R^n \mid \langle c^i(s), x\rangle \le 0 \ \ \forall i = 1, \cdots, k\},$$
$$X^s = X \setminus \mathrm{int}(X + C^s).$$

If the perturbation parameter s is small, then $X^s = X^\circ$ and the program (1) is equivalent to the following perturbated program

$$\text{minimize}\{\langle c, x\rangle, \ | x \in X^s\}. \tag{2}$$

A perturbation parameter s satisfying $X^s = X^\circ$ is said to be valid. It is known that this parameter is valid if it is positive and small enough. The program (2) is easier than (1) because of no duality gap. A valid perturbation parameter could be very small, and therefore it may cause numerical difficulty. In order to obtain a more computationally reliable algorithm, we use a lexicographic ordering approach. This well-known approach helps us to find an optimal solution without worrying about the specific value of the perturbation parameter.

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2 Dual Representations

For a given perturbation s we define

$$V^s = \left\{ v \in R^n \,\middle|\, \sup_{x \in X + C^s} \langle v, x\rangle \le 1 \right\},$$
$$g(v) = \inf\{\langle c, x\rangle \mid \langle v, x\rangle \ge 1,\ x \in X\}.$$

Then, V^s is a polytope and $g(\cdot)$ is a quasi-concave function. A dual program of the program (2) is [1]

$$\min\{g(v) \mid \ v \in V^s\}. \tag{3}$$

For a valid perturbation parameter s one has

$$\min\{g(v) \mid \ v \in V^s\} = \min(1). \tag{4}$$

In the sequel, we present another dual representation which is more convenient to interpret in term of lexicographic ordering.

For any $t > 0$ we define

$$U^t = \left\{ u = \sum_{i=1}^{k} \lambda_i c^i \,\middle|\, \sup_{x \in X} \langle u, x\rangle \le 1, \lambda_i \ge t \ \ \forall i = 1, \cdots, k \right\}, \tag{5}$$

getting the following dual program

$$\min\{g(u) | u \in U^t\}. \tag{6}$$

The dual (5)-(6) can be rewritten as

$$\left|\begin{array}{ll} \text{minimize} & G(\lambda), \\ \text{subject to} & \lambda \in \Lambda^t, \end{array}\right. \tag{7}$$

where for $\lambda \in \mathbb{R}^k$

$$G(\lambda) = \inf\left\{\langle c, x\rangle \,\middle|\, \left\langle \sum_{i=1}^{k} \lambda_i c^i, x\right\rangle \ge 1,\ x \in X \right\},$$
$$\Lambda^t = \left\{\lambda \in R^k \,\middle|\, \sum_{i=1}^{k} \lambda_i \langle c^i, x\rangle \le 1 \ \ \forall x \in X, \ \lambda_i \ge t \ \ \forall i = 1, \cdots, k\right\}. \tag{8}$$

$G(\cdot)$ is a quasi-concave function on $\mathbb{R}^k$, Λ^t is a polytope in $\mathbb{R}^k$. For $t > 0$ small enough one has

$$\min(7) = \min(1),$$

and if λ° solves (7) then any solution of

$$\min\left\{ \langle c, x\rangle \,\middle|\, \sum_{i=1}^{k} \lambda_i^\circ \langle c^i, x\rangle \ge 1,\ x \in X \right\}$$

solves (1).

3 Lexicographic Rule

We denote by Γ^t the set of vertices of Λ^t. Each vertex $\lambda \in \Gamma^t$ corresponds to a basic feasible solution and it can be represented as $\lambda = \alpha + t\beta$. The set of such vectors (α, β) is called the shadow of Γ^t. It has been known that there is $t_0 > 0$ such that for any $t \in (0, t_0]$ the system

$$\sum_{i=1}^{k} \lambda_i \langle c^i, x\rangle \leq 1 \quad \forall x \in Z, \lambda_i \geq t \quad \forall i = 1, \cdots, k$$

has the same basic feasible solutions, hence Γ^t has the same shadow Γ. Let Δ be the convex hull of Γ. The set Δ is called the shadow of Λ^t with $t > 0$ small enough. For each $(\alpha, \beta) \in \mathbf{R}^k \times \mathbf{R}^k$ we define $h(\alpha, \beta) \in \mathbf{R}^2$:

$$h(\alpha,\beta) = \underset{x \in X}{\text{lexico-min}} \left(\sum_{i=1}^{k} \alpha_i \langle c^i, x\rangle, \sum_{i=1}^{k} \beta_i \langle c^i, x\rangle \right). \quad (9)$$

Let us consider the following parametric LP

$$\min \left\{ \langle c, x\rangle \;\middle|\; \sum_{i=1}^{k} (\alpha_i + t\beta_i)\langle c^i, x\rangle \geq 1, x \in X \right\} \quad (10)$$

There is $t_0 > 0$ such that the program (10) has the same optimal simplex tableau and the optimal value $G(\alpha + t\beta)$ has the following form for any $t \in (0, t_0]$

$$G(\alpha+t\beta) = \begin{cases} \theta_1(\alpha,\beta) & \text{if } G(\alpha+t\beta) \text{ is constant of } t \in (0,t_0] \\ \theta_1(\alpha,\beta) + \dfrac{1}{\theta_2(\alpha,\beta) + t\theta_3(\alpha,\beta)} & \text{otherwise,} \end{cases}$$

where $\theta_3(\alpha, \beta) \neq 0$.

We define a function $F : \mathbf{R}^k \times \mathbf{R}^k \to \overline{\mathbf{R}} \times \overline{\mathbf{R}}$.

$$F(\alpha,\beta) = \begin{cases} (+\infty, 0) & \text{if } h(-\alpha,-\beta) \succ (-1,0) \\ (\theta_1(\alpha,\beta), 0) & \text{if } h(-\alpha,-\beta) \preceq (-1,0) \\ & \text{and } G(\alpha+t\beta) \text{ is constant of } t \in (0,t_0] \\ \left(\theta_1(\alpha,\beta) + \frac{1}{\theta_2(\alpha,\beta)}, -\frac{\theta_3(\alpha,\beta)}{\theta_2^2(\alpha,\beta)}\right) & \text{otherwise.} \end{cases}$$

The function $F(\cdot)$ is called the shadow of $G(\cdot)$.

Using the shadow Δ of Λ^t and the shadow F of G we can interpret the dual program (7) as follows

$$\text{lexico-minimize}\{F(\alpha,\beta) | (\alpha,\beta) \in \Gamma\}. \quad (11)$$

This program is called the shadow of the dual program (7). Since F is quasi-concave and Δ is the convex hull of Γ, an optimal solution of this problem is also a lexicographical minimizer of $F(\cdot)$ on Δ.

4 Relaxation Algorithm

Let Z_j be a subset of Z such that the relaxation Λ_j^t of Λ^t:

$$\Lambda_j^t \;:=\; \left\{ \lambda \in R^k \;\middle|\; \sum_{i=1}^{k} \lambda_i \langle c^i, x\rangle \leq 1 \ \forall x \in Z_j, \right.$$
$$\left. \lambda_i \geq t \ \forall i = 1, \cdots, k \right\}$$

is bounded. Γ_j^t is the set of vertices of Λ_j^t. Set $\Delta_j = \text{conv}(\Gamma_j)$. Δ_j is a relaxation of Δ. If $h(\alpha, \beta) \preceq (1, 0)$ for $(\alpha, \beta) \in \Gamma_j$, then $(\alpha, \beta) \in \Gamma$.

Relaxation Algorithm

Iteration j ($j = 1, 2, \cdots$) This iteration is entered with a subset Z_j of Z which defines Λ_j^t, and Γ_j of feasible basic solutions of Λ_j^t with $t > 0$ small enough.

Step a For any $(\alpha, \beta) \in \Gamma_j$ we compute $F(\alpha, \beta)$ by solving the parametric LP (10) in which t is supposed to be positive and small enough. If $F(\alpha, \beta) \prec (+\infty, 0)$, then denote by $x(\alpha, \beta)$ the optimal solution of (10) with $t > 0$ small enough.

Step b Define (α^j, β^j) such that

$$F(\alpha^j, \beta^j) = \text{lexico-min}\{F(\alpha,\beta) \mid (\alpha,\beta) \in \Gamma_j\}$$

Step c Solve

$$\underset{x \in X}{\text{lexico-min}} \left\{ \sum_{i=1}^{k} \alpha_i^j \langle c^i, x\rangle, \sum_{i=1}^{k} \beta_i^j \langle c^i, x\rangle \right\}$$

obtaining its optimal value $h(\alpha^j, \beta^j)$ and its optimal basic solution x^j.

Step d If $h(\alpha^j, \beta^j) \preceq (1, 0)$ then stop: $x(\alpha^j, \beta^j)$ is optimal to (1) , else update $Z_{j+1} = Z_j \cup x^j$.

Step e Define a new relaxation

$$\Lambda_{j+1}^t \;=\; \left\{ \lambda \in \mathbf{R}^k \;\middle|\; \sum_{i=1}^{k} \lambda_i \langle c^i, x\rangle \leq 1 \ \ \forall x \in Z_{j+1}, \right.$$
$$\left. \lambda_i \geq t \ \ \forall i = 1, \cdots, k \right\},$$

and compute Γ_{j+1}. Go to iteration $j + 1$.

If $h(\alpha^j, \beta^j) \preceq (1, 0)$ then $x(\alpha^j, \beta^j)$ does not depend on the parameter and it solves (1). The relaxation algorithm terminates after finitely many iteration yielding an optimal solution of the program (1).

References

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