Fractional Degree-two Polytopes and Ideal Polytopes of Bidirected Graphs*

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1. Introduction

A *bidirected graph* ([2]) $G = (V, A; \partial)$ is a graph with a vertex set V, an arc set A and a boundary operator $\partial : A \to \mathbb{Z}^V$, where for each arc $a \in A$ there exist $v, w \in V$ (called *end-vertices* of a) such that one of the following three holds:

(1) $\partial a = v + w$ (arc a has two tails at v and w),

(2) $\partial a = -v - w$ (arc a has two heads at v and w),

(3) $\partial a = v - w$ (arc a has a tail at v and a head at w).

Here, each $\partial a \in \mathbb{Z}^V$ is represented by an element of a free module with a base V. If $v = w$ in (1)~(3), then arc a is called a *selfloop*. For simplicity we do not allow any selfloop of type (3) in the following. See Figure 1.1 for an example of a bidirected graph with $V = \{1, 2, 3, 4\}$.

Recently, Ando, Fujishige and Nemoto [1] showed that the minimum-weight ideal problem on bidirected graphs can be reduced to the minimum-weight ideal problems for ordinary directed graphs.

On the other hand, the concept of degree-two inequalities is introduced by E. L. Johnson and M. W. Padberg [4]. They noticed that there exists a natural correspondence between bidirected graphs and degree-two inequalities.

An inequality of n variables $x_1, \cdots, x_n$ is called *degree-two* if it is either $x_i + x_j \le 1$, $-x_i - x_j \le -1$ or $x_i - x_j \le 0$ for some $i, j = 1, \cdots, n$. For example, the following is a system of degree-two inequalities.

$$-2x_1 \le -1, -x_1 + x_2 \le 0, x_2 + x_3 \le 1, 2x_3 \le 1,$$
$$x_3 + x_4 \le 1, x_3 - x_4 \le 0, x_2 - x_3 \le 0. \tag{1.1}$$

The 0-1 solutions of a degree-two inequalities are of special interest. The stable sets, the node covers, the ideals of a (directed) graph are described as the 0-1 solutions of systems of degree-two inequalities. It should be also noted that degree-two constraints are, in disguise, a complete set of implicants of length at most two ([3]).

In this paper, we consider a relaxation of the 0-1 solutions of degree-two inequalities, namely, we consider the solution set of degree-two inequalities and the inequalities $0 \le x_j \le 1$ $(j = 1, \cdots, n)$, which we call a fractional degree-two polytope.

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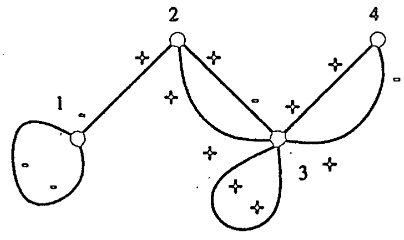

Figure 1.1: An Example of a Bidirected Graph.

2. Preliminaries

In an obvious way, one can associate a bidirected graph G to a system of degree-two inequalities (see [4]). For our example (1.1) the bidirected graph in Figure 1.1 corresponds. Now a system of degree two inequalities described in terms of a bidirected graph $G = (V, A)$ as

$$\langle \partial a, x \rangle \ \le \ \frac{1}{2} \langle \partial a, \mathbb{1}_V \rangle \quad (a \in A), \tag{2.1}$$

where $\mathbb{1}_V$ stands for the all-1 column vector, $\langle \cdot, \cdot \rangle$ is the (canonical) inner product, and ∂a should be regarded as a vector in $\mathbb{R}^V$. Conversely, given any bidirected graph $G = (V, A; \partial)$, the system (2.1) of inequalities is degree-two. Hence, from now on, we always associates a bidirected graph G with a system of degree-two inequalities.

Given a bidirected graph $G = (V, A; \partial)$, we call the solution set of the system

$$\langle \partial a, x \rangle \ \le \ \tfrac{1}{2} \langle \partial a, \mathbb{1}_V \rangle \quad (a \in A), \tag{2.2}$$
$$0 \le x(v) \le 1 \quad (v \in V) \tag{2.3}$$

of inequalities a *fractional degree-two polytope* associated with bidirected graph $G = (V, A; \partial)$ and denote it by FD2P(G).

An *ideal polytope* IP(G) associated with a bidirected graph $G = (V, A; \partial)$ is defined as the solution set of the system

$$\langle \partial a, x \rangle \le 0 \quad (a \in A), \tag{2.4}$$
$$-1 \le x(v) \le 1 \quad (v \in V) \tag{2.5}$$

of inequalities.

We denote by 3^V the set of all the ordered pair of disjoint subsets of V, i.e., $3^V = \{(X, Y) \mid X, Y \subseteq V, X \cap$

$Y = \emptyset\}$. We call each element of 3^V a *signed subset* of V. An integral solution x of (2.4)~(2.5) is made correspond to a signed subset (X, Y) of V as

$$(X,Y) = (\{v \mid v \in V, x(v) = 1\}, \{v \mid v \in V, x(v) = -1\}). \tag{2.6}$$

and is called an *ideal* of G. An ideal (X, Y) of G is called *spanning* if $X \cup Y = V$. Let us denote by $\mathcal{I}(G)$ the set of all the ideals of G.

Given a bidirected graph $G = (V, A; \partial)$ and a weight function $w : V \to \mathbf{R}$, the *minimum-weight ideal problem* ([1]) is defined as follows: Minimize$\{w(X) - w(Y) \mid (X, Y) \in \mathcal{I}(G)\}$.

It follows from the definitions that

Lemma 2.1: *For any bidirected graph* $G = (V, A; \partial)$ *we have* $x \in \mathrm{FD2P}(G)$ *if and only if* $2x - \mathbf{1}_V \in \mathrm{IP}(G)$. *Furthermore, for any* $w: V \to \mathbf{R}$ x *is an optimal solution for* $\min\{\sum_{v \in V} w(v)x(v) | x \in \mathrm{FD2P}(G)\}$ *if and only if* $2x - \mathbf{1}_V$ *is an optimal solution for* $\min\{\sum_{v\in V} w(v)x(v) \mid x \in \mathrm{IP}(G)\}$. □

For any subset U of vertex set V the *reflection of* $G = (V, A; \partial)$ *by* U is the bidirected graph $G' = (V, A; \partial')$ defined as follows. For each arc $a \in A$, if $\partial a = \pm v \pm w$, we define

$$\partial' a = \pm\epsilon(v)v \pm \epsilon(w)w, \tag{2.7}$$

where for each $v \in V$ $\epsilon(v) = 1$ if $v \notin U$ and $= -1$ if $v \in U$. We denote the reflection G' by $G{:}U$.

3. The Integrality of IP(G)'s

Given a bidirected graph $G = (V, A; \partial)$, the *signed covering graph* $\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial})$ of G is an ordinary directed graph defined as follows. The vertex set $\tilde{V}$ is given by $\tilde{V} = V \times \{+, -\}$ and the arc set $\tilde{A}$ by $\tilde{A} = \{a^{(+)} \mid a \in A\} \cup \{a^{(-)} | a \in A\}$. Moreover, the boundary operator $\tilde{\partial}$ in $\tilde{G}$ is defined as follows: For each $a \in A$, (i) if $\partial a = v - w$, then $\tilde{\partial} a^{(+)} = (v, +) - (w, +)$, $\tilde{\partial} a^{(-)} = (w, -) - (v, -)$; (ii) if $\partial a = v + w$, then $\tilde{\partial} a^{(+)} = (v, +) - (w, -)$, $\tilde{\partial} a^{(-)} = (w, +) - (v, -)$; (iii) if $\partial a = -v - w$, then $\tilde{\partial} a^{(+)} = (v, -) - (w, +)$, $\tilde{\partial} a^{(-)} = (w, -) - (v, +)$.

Suppose that we are given a bidirected graph $G = (V, A; \partial)$. Let us consider the linear programming problem (P_w) : Minimize$\{\sum_{v \in V} w(v)x(v) | x \in \mathrm{IP}(G)\}$, where $w : V \to \mathbf{R}$ is given weight function. Associated with Problem (P_w), we define a linear programming problem $(\widetilde{\mathrm{P}_w})$ as follows.

$$(\widetilde{\mathrm{P}_w}): \quad \mathrm{Min} \quad \sum_{v\in V} (\tilde{w}(v,+)\tilde{x}(v,+) + \tilde{w}(v,-)\tilde{x}(v,-))$$

$$\mathrm{s.t.} \quad \langle \tilde{\partial}\tilde{a}, \tilde{x} \rangle \le 0 \quad (\tilde{a} \in \tilde{A}), \tag{3.1}$$

$$0 \le \tilde{x}(v, \pm) \le 1 \quad (v \in V), \tag{3.2}$$

where $\tilde{w} : \tilde{V} \to \mathbf{R}$ is defined by $\tilde{w}(v,+) = w(v)$, $\tilde{w}(v,-) = -w(v)$ for $v \in V$.

For $x \in \mathrm{IP}(G)$ define $\tilde{x} \in \mathbf{R}^{\tilde{V}}$ by

$$\tilde{x}(v,+) = \max\{0, x(v)\}, \ \tilde{x}(v,-) = -\min\{0, x(v)\} \tag{3.3}$$

for each $v \in V$.

We call a vector $\tilde{z} \in \mathbf{R}^{\tilde{V}}$ *isotropic* if for each $v \in V$ $\tilde{z}(v,+)\tilde{z}(v,-) = 0$ holds.

Lemma 3.1: *The mapping defined by* (3.3) *gives a one-to-one correspondence between the set of the optimal solutions of* (P_w) *and the set of the isotropic optimal solutions of* $(\widetilde{\mathrm{P}_w})$. □

Theorem 3.2: *For any bidirected graph* G IP(G) *is integral and* FD2P(G) *is half-integral.* □

Corollary 3.3: *For any bidirected graph* G *the linear programming problem over* FD2P(G) *can be reduced to the minimum-weight ideal problem for* G, *and vice versa.* □

4. Characterizations of Integral FD2P(G)

A bidirected graph $G = (V, A; \partial)$ is called *balanced* if for some $U \subseteq V$ the reflection $G{:}U$ of G by U is an ordinary directed graph.

Theorem 4.1: FD2P(G) *is integral if and only if* G *is balanced.* □

For any $Q \subseteq [0,1]^V$ and $U \subseteq V$ define the *negation* $Q!U$ of Q at U by $Q!U = \{x!U | x \in Q\}$, where $x!U$ is defined by $x\,!\,U(v) = 1 - x(v)$ if $v \in U$ and $= x(v)$ otherwise for each $v \in V$.

Corollary 4.2: *For any bidirected graph* G FD2P(G) *is integral if and only if* FD2P(G) ! U *is an ordinary ideal polytope for some* $U \subseteq V$. □

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