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Pairwise-Bargained Consistency: Relationship between Nucleolus and τ-value through an Associated Bimatrix Game

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0. Introduction We introduce pairwise-bargained consistency with a reference point, and use as reference points the maxmin and the minmax value within pure strategies of a certain constant-sum bimatrix game, and also the game value within mixed strategies of it. We show that the pairwise-bargained consistency with reference point being the maxmin or the minmax value determines the nucleolus in some class of transferable utility games. (This result is known in the bankruptcy games and the pseudo-concave games with respect to supersets of the managers.) This class of games whose element we call a pseudo-concave game with respect to essential coalitions, of course, includes the bankruptcy games and the pseudo-concave games with respect to supersets of the managers. It is proved that this class of games is exactly the same as the class of games which have a nonempty core that is determined only by one-person and $(n-1)$-person coalition constraints. We interpret the τ-value of a quasibalanced transferable utility game by the pairwise-bargained consistency with reference point being the game value. By combining these results, if a transferable utility game in this class is also semiconvex, then the nucleolus and the τ-value are characterized by the pairwise-bargained consistency with different reference points which are given by the associated bimatrix game.

1. Pairwise-bargained consistency Let us consider the following situation:
Situation 1: Player i and j have to divide the total amount X. Player $i(j)$ can claim $r_{ij}(X)(r_{ji}(X))$ against player $j(i)$ respectively. We assume that $X \in \mathbb{R}$, and $r^{\{i,j\}}(X) := (r_{ij}(X), r_{ji}(X))$ where $r_{ij} : \mathbb{R} \to \mathbb{R}$ and $r_{ji} : \mathbb{R} \to \mathbb{R}$.

Natural division $ND^{r^{\{i,j\}}}(X)$ of the amount X between players i and j with reference point r.

1. Player i and j firstly get $r_{ij}(X)$ and $r_{ji}(X)$ respectively.
2. Then they divide the surplus of the total amount equally.

We say $x^{\{i,j\}} = (x_i, x_j) \in \mathbb{R}^2$ is *bargained consistent with reference point* r if $ND^{r^{\{i,j\}}}(x_i + x_j) = x^{\{i,j\}}$, because there is no incentive for the players to reallocate the total amount $x_i + x_j$ between them.

Let us consider the following slightly different situation:
Situation 2: Any player k of a couple i and j estimates that the smallest and largest amount which he can get are $f(k)$ and Δ_k respectively. Both player i and j are expected to receive the amount x_i and x_j respectively where $f(k) \le x_k \le \Delta_k$ for $k = i, j$. Should they accept the pair of allocations (x_i, x_j)? Or are there any incentive to reallocate the total amount $X = x_i + x_j$ between them?

In order to convert Situation 2 to Situation 1 we offer several ways to determine reference points by using an associated bimatrix in Table 1. Note that since the game is constant-sum, every Nash equilibrium gives each player his unique expected payoff. We denote the pair of game values (the pair of the expected payoffs of both players for any Nash equilibria) by $(e_{ij}(x), e_{ji}(x))$. Let us define $c_{ij}(x)$ and $d_{ij}(x)$ to be the maxmin and minmax value (within pure strategies) of player i's part of the above bimatrix, and similarly $c_{ji}(x)$ and $d_{ji}(x)$ to be the maxmin and minmax value of player j's part of the above bimatrix.
Let $x \in \mathbb{R}^N$ be such that $f(i) \le x_i \le \Delta_i$ for all $i \in N$ and $r = (r_{ij})_{(i,j)\in N\times N, i\neq j}$

TABLE 1. Associated bimatrix game

where $f(i) + f(j) \le X \le \Delta_i + \Delta_j$

		player j	
		Y(ou)	I
player i	C	$f(i), X - f(i)$	$X - f(j), f(j)$
	B	$\Delta_i, X - \Delta_i$	$X - \Delta_j, \Delta_j$

where $r_{ij} : \mathbb{R} \to \mathbb{R}$. We say that x is *pairwise-bargained consistent with reference point* r if $x^{\{i,j\}} = (x_i, x_j)$ is bargained consistent with reference point r for all $i, j \in N$ and $i \neq j$, that is, $ND^{r^{\{i,j\}}}(x_i + x_j) = x^{\{i,j\}}$ for all $i, j \in N$ and $i \neq j$. We use as reference points three points c, d, and e. We also use the term *pairwise-modest(medium, greedy)-bargained consistent* in place of pairwise-bargained consistent with reference point $c(e, d)$ respectively.

2. Pseudo-concave game with respect to essential coalitions Let (N, v) be a transferable utility game (TU-game) where N is the set of players and $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. We call $x \in \mathbb{R}^N$ an *allocation* if $\sum_{i \in N} x_i = v(N)$. Let us call a coalition S *essential* if $v(S) > \sum_{i \in S} v(\{i\})$. Also call a TU-game (N, v) *pseudo-concave w.r.t. essential coalitions* if it satisfies

$$v(S) \le \max \left[\sum_{i \in S} f(i), v(N) - \sum_{i \in N-S} \Delta_i \right] \quad \text{for all } S \subset N, S \neq \emptyset,$$

$$f(i) \le \Delta_i \quad \text{for all } i \in N, \quad \text{and} \quad \sum_{i \in N} f(i) \le v(N) \le \sum_{i \in N} \Delta_i .$$

Corollary 2.2. *A TU-game (N, v) has a nonempty core which is determined only by one-person and $(n-1)$-person coalition constraints if and only if it is pseudo-concave w.r.t. essential coalitions.*

3. Nucleolus: pairwise-modest(or greedy)-bargained consistency

Theorem 3.2. *Let $f(i) := v(\{i\})$ and $\Delta_i := v(N) - v(N - \{i\})$ where (N, v) is a TU-game and $i \in N$.*
If (N, v) is pseudo-concave w.r.t. essential coalitions, then the pairwise-modest(or greedy)-bargained consistency determines a unique allocation which coincides with the nucleolus $\eta(v)$.

4. Relationship between nucleolus and τ-value

Theorem 4.2. *Let $f(i) := \Delta_i - \lambda_i^v$ and $\Delta_i := v(N) - v(N - \{i\})$, where (N, v) is a TU-game and $i \in N$. If (N, v) is quasibalanced, then the τ-value $\tau(v)$ is a unique pairwise-medium-bargained consistent allocation.*

Corollary 4.3. *Let $f(i) := v(\{i\})$ and $\Delta_i := v(N) - v(N - \{i\})$, where (N, v) is a TU-game and $i \in N$.*
If (N, v) is pseudo-concave w.r.t. essential coalitions and semiconvex, then the nucleolus $\eta(v)$ agrees with a unique pairwise-modest(or greedy)-bargained consistent allocation, whereas the τ-value $\tau(v)$ with a unique pairwise-medium-bargained consistent allocation.