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Further Results for Multiclass M/G/1 Queues with Feedback (I)[1]

01505150 University of Tsukuba 平山哲治 HIRAYAMA Tetsuji

1 Introduction.

We consider multiclass M/G/1 queues with feedbacks. Customers in the system are classified into J groups. Further group i at station i consists of L_i classes of customers ($i = 1, \ldots, J$). A single server serves customers at these stations. Customers are served according to a predetermined scheduling algorithm. After receiving a service, each customer feedbacks to the system or departs from the system.

We consider priority scheduling algorithms in which priority levels are assigned to the stations (groups) such that a station with lower number has a higher priority level. All customers belonging to the same group have the same priority level. Service order of customers in a group is FCFS. Once a customer begins a service, his service is not interrupted by any other customer until he completes his current service stage (non-preemptive).

2 Formulation of the model.

A class α customer at station i is called an (i, α)-*customer*. Let $\mathcal{S} \equiv \{(i, \alpha) : i = 1, \ldots, J \text{ and } \alpha = 1, \ldots, L_i\}$, and $J_c \equiv \sum_{i=1}^{J} L_i$. Class (i, α)-customers arrive from outside the system according to an independent Poisson process with rate $\lambda_{i\alpha}$ ($(i, \alpha) \in \mathcal{S}$). Let $\lambda_i \equiv \sum_{\alpha=1}^{L_i} \lambda_{i\alpha}$ and $\lambda \equiv \sum_{i=1}^{J} \lambda_i$. Service times $S_{i\alpha}$ of (i, α)-customers are independently and arbitrarily distributed with mean $E[S_{i\alpha}]$ and second moment $\overline{s^2}_{i\alpha}$. After receiving a service, an (i, α)-customer either feeds back to the system and is changed into a (j, β)-customer with probability $p_{i\alpha, j\beta}$, or departs from the system with probability $p_{i\alpha,00}$ ($(i, \alpha), (j, \beta) \in \mathcal{S}$) (For convenience, $p_{00,j\beta} \equiv 0$). Let $\mathbf{P} \equiv (p_{i\alpha,j\beta} : (i, \alpha), (j, \beta) \in \mathcal{S}) \in \mathcal{R}^{J_c \times J_c}$.

Let $T_{i\alpha,j}$ be a total amount of service times a customer receives until he departs from the system or enters one of stations $j+1, \ldots, J$ for the first time after completing his current service stage as an (i, α)-customer ($(i, \alpha) \in \mathcal{S}$). Let $\overline{T}_{i\alpha,j}$ be its expected value and $\overline{T}_{i\alpha,j}(r)$ be its expected value conditioned on his current remaining service time r at station i. Then

$$\begin{aligned} \overline{T}_{i\alpha,j} &= E[S_{i\alpha}] + \textstyle\sum_{k=1}^{j} \sum_{\gamma=1}^{L_k} p_{i\alpha,k\gamma} \overline{T}_{k\gamma,j}, \quad j = 0, 1, \ldots, J, \\ \overline{T}_{i\alpha,j}(r) &= r + \textstyle\sum_{k=1}^{j} \sum_{\gamma=1}^{L_k} p_{i\alpha,k\gamma} \overline{T}_{k\gamma,j}, \quad j = 0, 1, \ldots, J, \end{aligned} \tag{2.1}$$

for $(i, \alpha) \in \mathcal{S}$ ($\overline{T}_{00,j} \equiv 0$). Further we define

$$\rho_j^+ \equiv \textstyle\sum_{i=1}^{j} \sum_{\alpha=1}^{L_i} \lambda_{i\alpha} \overline{T}_{i\alpha,j}, \quad j = 1, \ldots, J, \tag{2.2}$$

($\rho_0^+ \equiv 0$). Let us consider the following assumption:

Assumption 1. $\mathbf{P}^n \to 0$ as $n \to \infty$, and $\rho_J^+ < 1$. □

Let (ι, a) denote the station-class pair of a customer being served currently, and let r denote his remaining service time. Number of (i, α)-customers in the system, who are not being served, is denoted by $n_{i\alpha}$. Let $\mathbf{n}_{i\cdot} \equiv (n_{i\alpha} : \alpha = 1, \ldots, L_i)$, $\mathbf{n} \equiv (\mathbf{n}_{1\cdot}, \ldots, \mathbf{n}_{J\cdot})$, and $n_i \equiv \sum_{\alpha=1}^{L_i} n_{i\alpha}$ ($i = 1, \ldots, J$). Let v_{im} be a remaining service time of a group i customer at the m^{th} position of its queue. Further let $v_i \equiv \sum_{m=1}^{n_i} v_{im}$, which is often called a *work* at station i ($i = 1, \ldots, J$). The class of a customer at the m^{th} position of station i is denoted by $c_i(m)$. We define a vector $(\mathbf{v}, \mathbf{n}) \equiv (v_1, \ldots, v_J, n_{11}, \ldots, n_{JL_J})$.

Let us consider the e^{th} customer ($\mathbf{c}^e$) arrives from outside the system at one of the stations at some epoch σ_0^e ($e = 1, 2, \ldots$). Then let σ_k^e be a time epoch just when he would arrive at one of the stations after completing his k^{th} service. Let us consider transition epochs of the system consist of customer arrival epochs and service completion epochs. Then let $(K(t), \Gamma(t)) \in \mathcal{S}$ denote the station-class pair of a customer arrived at the last transition epoch before t ($t \geq 0$). $(K(t), \Gamma(t)) = (0, 0)$ if a customer departs from the system at the transition epoch. The station-class pair of a customer being served at time t is denoted by $(\iota(t), a(t))$, and his remaining service time at time t is denoted by $r(t)$. We assume that $(\iota(\tau), a(\tau)) = (0, 0)$ if the system is empty at time τ, or if τ is a service completion epoch. Then let $\mathcal{S}_0 \equiv \mathcal{S} \cup \{(0, 0)\}$. Number of (i, α)-customers in the system at time t is denoted by $n_{i\alpha}(t)$. Let $\mathbf{n}_{i\cdot}(t) \equiv (n_{i\alpha}(t) : \alpha = 1, \ldots, L_i)$, $\mathbf{n}(t) \equiv (\mathbf{n}_{1\cdot}(t), \ldots, \mathbf{n}_{J\cdot}(t))$, and $n_i(t) \equiv \sum_{\alpha=1}^{L_i} n_{i\alpha}(t)$, ($i = 1, \ldots, J$). Further let $v_{im}(t)$ be a remaining service time of a group i customer at the m^{th} position of its queue at time t. Further let $v_i(t) \equiv \sum_{m=1}^{n_i(t)} v_{im}(t)$, $\mathbf{V}(t) \equiv (v_{im}(t) : i = 1, \ldots, J \text{ and } m = 1, 2, \ldots)$, and $\mathbf{v}(t) \equiv (v_1(t), \ldots, v_J(t))$. Then we consider the stochastic process $\mathcal{Q} \equiv \{\mathbf{Y}(t) \equiv (K(t), \Gamma(t), \iota(t), a(t), r(t), \mathbf{V}(t), \mathbf{n}(t)) : t \geq 0\}$. Possible values of $\mathbf{Y}(t)$ ($t \geq 0$) are called *states* whose generic values are denoted by $\mathbf{Y} \equiv (k, \gamma, \iota, a, r, \mathbf{V}, \mathbf{n})$. The state space

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is denoted by $\mathcal{E}$.

We would like to derive two types of cost functions defined below. First type of the cost functions represents the mean sojourn times for each class of customers. For $e = 1, 2, \ldots$, we define

$$C^e_{Wi\alpha}(t) \equiv \begin{cases} 1, & \text{if } \mathbf{c}^e \text{ is an } (i,\alpha)\text{-customer at } t, \\ 0, & \text{if } \mathbf{c}^e \text{ is not an } (i,\alpha)\text{-customer at } t, \end{cases} \qquad (2.3)$$

for $(i, \alpha) \in \mathcal{S}$ $(t \geq 0)$. Then we define

$$W^e_{i\alpha} \equiv \int_0^{\infty} C^e_{Wi\alpha}(t)dt, \qquad (2.4)$$

$$W_{i\alpha}(\mathbf{Y}, e, l) \equiv E\left[\int_{\sigma^e_l}^{\infty} C^e_{Wi\alpha}(t)dt|\mathbf{Y}(\sigma^e_l) = \mathbf{Y}\right], \qquad (2.5)$$

for $(i, \alpha) \in \mathcal{S}$ and $l = 0, 1, \ldots$, where $\mathbf{Y} = (j, \beta, \iota, a, r, \mathbf{V}, \mathbf{n}) \in \mathcal{E}$ is the state of the system at time σ^e_l. Let us consider $\mathbf{c}^e$ is arrived at the system as a (j, β)-customer at time σ^e_l. His *initial stay* denotes a period from time σ^e_l until he completes his first service at station j just prior to σ^e_{l+1}. The length of his initial stay is called the initial sojourn time. Then for $\mathbf{c}^e$, we define

$$W^I_{i\alpha}(\mathbf{Y}, e, l) \equiv E\left[\int_{\sigma^e_l}^{\sigma^e_{l+1}} C^e_{Wi\alpha}(t)dt|\mathbf{Y}(\sigma^e_l) = \mathbf{Y}\right], \qquad (2.6)$$

for $(i, \alpha) \in \mathcal{S}$ and $l = 0, 1, \ldots$, where $\mathbf{Y} = (j, \beta, \iota, a, r, \mathbf{V}, \mathbf{n}) \in \mathcal{E}$. Trivially, $W^I_{i\alpha}(\mathbf{Y}, e, l) \equiv 0$ for $(i, \alpha) \neq (j, \beta)$. Then we have

$$W_{i\alpha}(\mathbf{Y}, e, l) = W^I_{i\alpha}(\mathbf{Y}, e, l) + E[W_{i\alpha}(\mathbf{Y}(\sigma^e_{l+1}), e, l+1)|\mathbf{Y}(\sigma^e_l) = \mathbf{Y}], \qquad (2.7)$$

for $(i, \alpha) \in \mathcal{S}$, $\mathbf{Y} \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$.

Second type of the cost functions is related to the expected work at each station. Let $C^e_{Gi\alpha}(t)$ be the remaining service time of $\mathbf{c}^e$ at time t in his current service stage when he is an (i, α)-customer at t. Then we define

$$G^e_{i\alpha} \equiv \int_0^{\infty} C^e_{Gi\alpha}(t)dt, \qquad (2.8)$$

$$G_{i\alpha}(\mathbf{Y}, e, l) \equiv E\left[\int_{\sigma^e_l}^{\infty} C^e_{Gi\alpha}(t)dt|\mathbf{Y}(\sigma^e_l) = \mathbf{Y}\right], \qquad (2.9)$$

$$G^I_{i\alpha}(\mathbf{Y}, e, l) \equiv E\left[\int_{\sigma^e_l}^{\sigma^e_{l+1}} C^e_{Gi\alpha}(t)dt|\mathbf{Y}(\sigma^e_l) = \mathbf{Y}\right], \qquad (2.10)$$

for $(i, \alpha) \in \mathcal{S}$ and $l = 0, 1, \ldots$, where $\mathbf{Y} \in \mathcal{E}$. Trivially, $G^I_{i\alpha}(\mathbf{Y}, e, l) \equiv 0$ for $(i, \alpha) \neq (j, \beta)$. Then we have

$$G_{i\alpha}(\mathbf{Y}, e, l) = G^I_{i\alpha}(\mathbf{Y}, e, l) + E[G_{i\alpha}(\mathbf{Y}(\sigma^e_{l+1}), e, l+1)|\mathbf{Y}(\sigma^e_l) = \mathbf{Y}], \qquad (2.11)$$

for $(i, \alpha) \in \mathcal{S}$, $\mathbf{Y} \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$.

3 Busy period processes.

Let us consider the system in state $\mathbf{Y} = (j, \beta, \iota, a, r, \mathbf{V}, \mathbf{n}) \in \mathcal{E}$ at some transition epoch. We select a set $\mathcal{C} = \mathcal{C}(\mathbf{Y})$ composed of some customers who stay in the system at that time.

For any $\mathbf{Y} \in \mathcal{E}$ and any $\mathcal{C} = \mathcal{C}(\mathbf{Y})$, let $B^j(\mathbf{Y}; \mathcal{C})$ be a (generalized) busy period of the system initiated with state $\mathbf{Y}$ until the first time when the system is cleared of the customers in $\mathcal{C}$ and customers at stations $1, \ldots, j$, except for customers who are initially in the system and are not in $\mathcal{C}$. We call $B^j(\mathbf{Y}; \mathcal{C})$ a *group j busy period* initiated with $(\mathbf{Y}; \mathcal{C})$. $B^0(\mathbf{Y}; \mathcal{C})$ denotes a time to complete services of the customers in $\mathcal{C}$. Then we have

$$E[B^j(\mathbf{Y}; \mathcal{C})] = (\sum\sum_{(i,m)\in\mathcal{C}} \overline{T}_{ic_\iota(m),j}(v_{im}))/(1 - \rho^+_j), \qquad (3.1)$$

$j = 1, \ldots, J$, where $v_{\iota 0} = r$ if $\mathcal{C}$ contains a group ι customer being served ($E[B^0(\mathbf{Y}; \mathcal{C})] = \sum\sum_{(i,m)\in\mathcal{C}} v_{im}$).

Let $N^j_{l\delta}$ and $V^j_{l\delta}$ respectively denote the number of (l, δ)-customers and the work of (l, δ)-customers at a completion epoch of a group j busy period ($0 \leq j < l \leq J$ and $(l, \delta) \in \mathcal{S}$). If the busy period is initiated by a (k, γ)-customer, their expected values are denoted by $\overline{N}^j_{k\gamma,l\delta}$ and $\overline{V}^j_{k\gamma,l\delta}$, respectively ($(k, \gamma) \in \mathcal{S}$). Then for any $(k, \gamma) \in \mathcal{S}$,

$$\overline{N}^j_{k\gamma,l\delta} = \lambda_{l\delta} E[S_{k\gamma}] + p_{k\gamma,l\delta} + \sum_{i=1}^{j} \sum_{\beta=1}^{L_\iota} \{\lambda_{i\beta} E[S_{k\gamma}] + p_{k\gamma,i\beta}\} \overline{N}^j_{i\beta,l\delta},$$

$$\overline{V}^j_{k\gamma,l\delta} = \{\lambda_{l\delta} E[S_{k\gamma}] + p_{k\gamma,l\delta}\} E[S_{l\delta}] + \sum_{i=1}^{j} \sum_{\beta=1}^{L_\iota} \{\lambda_{i\beta} E[S_{k\gamma}] + p_{k\gamma,i\beta}\} \overline{V}^j_{i\beta,l\delta},$$

for $0 \leq j < l \leq J$ and $(l, \delta) \in \mathcal{S}$. Now we define

$$\begin{aligned} &\xi^j_{l\delta} \equiv \lambda_{l\delta} + \textstyle\sum_{i=1}^{j} \sum_{\beta=1}^{L_\iota} \lambda_{i\beta} \overline{N}^j_{i\beta,l\delta}, \\ &\chi^j_{00,l\delta} \equiv 0, \\ &\chi^j_{k\gamma,l\delta} \equiv p_{k\gamma,l\delta} + \textstyle\sum_{i=1}^{j} \sum_{\beta=1}^{L_\iota} p_{k\gamma,i\beta} \overline{N}^j_{i\beta,l\delta}, \quad (k, \gamma) \in \mathcal{S}, \\ &\overline{\xi}^j_{l\delta} \equiv \xi^j_{l\delta} E[S_{l\delta}], \\ &\overline{\chi}^j_{k\gamma,l\delta} \equiv \chi^j_{k\gamma,l\delta} E[S_{l\delta}], \quad (k, \gamma) \in \mathcal{S}_0, \end{aligned} \qquad (3.2)$$

for $0 \leq j < l \leq J$ and $(l, \delta) \in \mathcal{S}$.

For any $\mathbf{Y} \in \mathcal{E}$ and any $\mathcal{C} = \mathcal{C}(\mathbf{Y})$, let $\overline{N}^j_{l\delta}(\mathbf{Y}; \mathcal{C})$ and $\overline{V}^j_{l\delta}(\mathbf{Y}; \mathcal{C})$ ($0 \leq j < l \leq J$ and $(l, \delta) \in \mathcal{S}$) be respectively the number of (l, δ)-customers and the work of (l, δ)-customers at a completion epoch of $B^j(\mathbf{Y}; \mathcal{C})$. Then we obtain

$$\overline{N}^j_{l\delta}(\mathbf{Y}; \mathcal{C}) = \sum_{m\in C_{l\delta}(\mathcal{C})} 1 + \sum\sum_{(i,m)\in\mathcal{C}} \left\{v_{im}\xi^j_{l\delta} + \chi^j_{ic_\iota(m),l\delta}\right\}, \qquad (3.3)$$

$$\overline{V}^j_{l\delta}(\mathbf{Y}; \mathcal{C}) = \sum_{m\in C_{l\delta}(\mathcal{C})} v_{lm} + \sum\sum_{(i,m)\in\mathcal{C}} \left\{v_{im}\overline{\xi}^j_{l\delta} + \overline{\chi}^j_{ic_\iota(m),l\delta}\right\}, \qquad (3.4)$$

where $C_{l\delta}(\mathcal{C}) = \{m : (l, m) \notin \mathcal{C}, c_l(m) = \delta\}$.

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