1997年度日本オペレーションズ・リサーチ学会
秋季研究発表会

Finding the Nucleolus of Assignment Games

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1. Introduction

The nucleolus is a single-point solution concept introduced by Schmeidler in [1]. This paper considers a problem for finding the nucleolus of assignment games. Assignment games with side payments are models of two-sided markets introduced by Shapley and Shubik in [2]. Recently, Solymosi and Raghavan [3] proposed an $O(v^3u)$ time algorithm for finding the nucleolus of an (u, v)-person assignment game with $v \leq u$. We show a characterization of the nucleolus of assignment games, which gives an idea to reduce the size of the problem when $u >> v$. The time complexity of our algorithm is $O(v^2u + v^4)$.

2. Preliminaries

For any pair of sets S_1, S_2, $B(S_1, S_2)$ denotes the complete bipartite graph with vertex sets S_1, S_2. An edge set M of $B(S_1, S_2)$ is called a *matching* when every vertex is incident to at most one member in M.

Assignment games are the model of two-sided markets, which consist of two disjoint sets of players denoted by U and V. The set of players $U \cup V$ is denoted by P. For each mixed pair of players $(i, j) \in U \times V$, there is associated a non-negative profit a_{ij}. The *characteristic function* $v : 2^P \rightarrow \mathbf{R}$ of the assignment game is defined as follows; (1) if $S \cap U = \emptyset$ or $S \cap V = \emptyset$, then $v(S) = 0$, (2) otherwise, $v(S)$ is equal to the weight of maximum weight matching in the graph $B(S \cap U, S \cap V)$ w.r.t. the edge-weight $\boldsymbol{a}$. In this paper, we denote the assignment game by the triplet $(U, V, \boldsymbol{a})$. For any *payoff vector* $\boldsymbol{x} \in \mathbf{R}^P$, the *excess* of a coalition $S \subseteq P$ is denoted by $e(S, \boldsymbol{x})$. The *excess vector* $\boldsymbol{e}(\boldsymbol{x})$ with respect to a payoff $\boldsymbol{x}$ is a vector indexed by 2^P whose entries are the excess $e(S, \boldsymbol{x})$, $\forall S \subset P$. The core of the assignment game $(U, V, \boldsymbol{a})$ is denoted by $\mathcal{C}(U, V, \boldsymbol{a})$. For any vector $\boldsymbol{w}$ indexed by a finite set S, $\theta\boldsymbol{w}$ denotes the sequence of elements in $\boldsymbol{w}$ whose entries are arranged in a non-increasing order. We denote the lexicographic order by $\geq_{\text{lex}}$ or $\leq_{\text{lex}}$. The nucleolus of the assignment game is the imputation $\boldsymbol{x}$ satisfying the condition that for any imputation $\boldsymbol{x}'$, $\theta\boldsymbol{e}(\boldsymbol{x}) \leq_{\text{lex}} \theta\boldsymbol{e}(\boldsymbol{x}')$.

3. The nucleolus of assignment games

We show a characterization of the nucleolus of the assignment game. The family of singleton coalitions $\{\{i\} \mid i \in P\}$ is denoted by P_1 and the family of mixed pair coalitions $\{\{i, j\} \mid (i, j) \in U \times V\}$ is denoted by P_2. For any payoff vector $\boldsymbol{x}$, $\boldsymbol{e}_{12}(\boldsymbol{x})$ denotes the vector indexed by $P_1 \cup P_2$ whose entries are excess $e(S, \boldsymbol{x})$ for all $S \in P_1 \cup P_2$.

Theorem 1 A payoff vector $\boldsymbol{x} \in \mathcal{C}(U, V, \boldsymbol{a})$ is the nucleolus if and only if $\forall \boldsymbol{x}' \in \mathcal{C}(U, V, \boldsymbol{a})$, $\theta\boldsymbol{e}_{12}(\boldsymbol{x}) \leq_{\text{lex}} \theta\boldsymbol{e}_{12}(\boldsymbol{x}')$.

In the rest of this paper, we assume that $|U| \geq |V|$. Let $\overline{M}$ be a matching in $B(U, V)$ which attains the value $v(P)$ and covers V. Let $\overline{U} \subseteq U$ be the set of vertices which are covered by $\overline{M}$. Now we construct a new assignment game, called the *auxiliary assignment game* (with respect to $\overline{M}$), as follows. We introduce two artificial players i_0, j_0 and we denote the set of players $\overline{U} \cup \{i_0\}$ by U^* and $V \cup \{j_0\}$ by V^*. The auxiliary assignment game is the assignment game with the set of players $U^* \cup V^*$ and the profit vector

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$\boldsymbol{b}$ indexed by $U^* \times V^*$ defined by

$$\boldsymbol{b}_{ij} = \begin{cases} a_{ij} & (\text{if } (i,j) \in \overline{U} \times V), \\ 0 & (\text{if } j = j_0), \\ \max\{a_{i'j} \mid i' \in U \setminus \overline{U}\} \\ & (\text{if } i = i_0 \text{ and } j \in V). \end{cases}$$

In the above definition, we assume that $\max\{a_{i'j} \mid i' \in U \setminus \overline{U}\} = 0$, when $\overline{U} = U$. The auxiliary assignment game is the assignment game defined by the triplet $(U^*, V^*, \boldsymbol{b})$. We denote the set of players $U^* \cup V^*$ by P^*. The characteristic function of $(U^*, V^*, \boldsymbol{b})$ is denoted by $v^* : 2^{P^*} \to \mathbf{R}$.

For any payoff vector $\boldsymbol{x}$ in $\mathcal{C}(U, V, \boldsymbol{a})$, the payoff vector $\boldsymbol{f}^{\mathrm{a}}(\boldsymbol{x})$ of the auxiliary game is defined by

$$\boldsymbol{f}^{\mathrm{a}}(\boldsymbol{x})_i = \begin{cases} 0 & \text{if } i = i_0 \text{ or } i = j_0, \\ x_i & \text{otherwise,} \end{cases}$$

and for any payoff vector $\boldsymbol{y}$ in $\mathcal{C}(U^*, V^*, \boldsymbol{b})$, the payoff vector $\boldsymbol{f}^{\circ}(\boldsymbol{y})$ of the original game is defined by

$$\boldsymbol{f}^{\circ}(\boldsymbol{y})_i = \begin{cases} y_i & \text{if } i \in (U \cup V) \setminus \overline{U}, \\ 0 & \text{if } i \in \overline{U}. \end{cases}$$

Then we have the following property.

Lemma 2
If $\boldsymbol{x} \in \mathcal{C}(U, V, \boldsymbol{a})$, then $\boldsymbol{f}^{\mathrm{a}}(\boldsymbol{x}) \in \mathcal{C}(U^*, V^*, \boldsymbol{b})$. If $\boldsymbol{y} \in \mathcal{C}(U^*, V^*, \boldsymbol{b})$, then $\boldsymbol{f}^{\circ}(\boldsymbol{y}) \in \mathcal{C}(U, V, \boldsymbol{a})$.

For any payoff vector $\boldsymbol{y} \in \mathbf{R}^{P^*}$ and for any coalition $S \subset P^*$, the corresponding excess and the excess vector of the auxiliary assignment game is denoted by $e^*(S, \boldsymbol{y})$ and $\boldsymbol{e}^*(\boldsymbol{y})$, respectively. The family of singleton coalitions of P^* is denoted by P_1^* and the family of mixed pair coalitions is denoted by P_2^*. For any payoff vector $\boldsymbol{y}$, $\boldsymbol{e}_{12}^*(\boldsymbol{y})$ and $e_2^*(\boldsymbol{y})$ denote the vectors obtained by restricting $e^*(\boldsymbol{y})$ to $P_1^* \cup P_2^*$ and P_2^*, respectively.

Theorem 3 A payoff $\boldsymbol{y} \in \mathcal{C}(U^*, V^*, \boldsymbol{b})$ is the nucleolus of the auxiliary assignment game if and only if $\theta \boldsymbol{e}_2^*(\boldsymbol{y}) \leq_{\text{lex}} \theta \boldsymbol{e}_2^*(\boldsymbol{y}')$, $\forall \boldsymbol{y}' \in \mathcal{C}(U^*, V^*, \boldsymbol{b})$.

Theorem 4 A payoff $\boldsymbol{y} \in \mathcal{C}(U^*, V^*, \boldsymbol{b})$ is the nucleolus of the auxiliary assignment game if and only if $\boldsymbol{f}^{\circ}(\boldsymbol{y})$ is the nucleolus of the original assignment game.

4. Algorithm

By applying Theorem 4, we can reduce the size of the original problem when $u = |U| >> |V| = v$. Then we can construct an $\mathrm{O}(u^2 v + v^4)$ time algorithm by employing the algorithm by Solymosi and Raghavan [3] as follows. First, we find a maximum weight matching $\overline{M}$ by ordinary Hungarian method in $\mathrm{O}(v^2 u)$ time. Next, we construct the auxiliary assignment game $(U^*, V^*, \boldsymbol{b})$ in $\mathrm{O}(u + v)$ time. Lastly, we find the nucleolus $\boldsymbol{y}$ of the auxiliary assignment game by Solymosi and Raghavan's algorithm in $\mathrm{O}((v+1)^4) = \mathrm{O}(v^4)$ and output the payoff vector $\boldsymbol{f}^{\circ}(\boldsymbol{y})$ as the nucleolus of original assignment game. The overall complexity is $\mathrm{O}(v^2 u + v^4)$ time.

References

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