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Social Welfare Analysis of Balking System with Heterogeneous Users and Optimal Externality Pricing

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1. Introduction Since the middle of 80's, the economic analysis of queueing systems has been drawing researchers' attention in the field of MIS study as well as Operations Management. In this paper, we focus on the social welfare analysis of a balking system with heterogeneous users. As Naor (1969) pointed out, the queueing system always involves a built-in inefficiency, called the congestion externality. We investigate the relationships between the social welfare and the users' demand function under the existence of the externality. We also investigate the role of price mechanism as a control strategy for the externality.

2. Balking System with Heterogeneous Users We consider a single server queue with exponential server with service rate $\mu > 0$. Customers arrive at the system in a Poisson stream with parameter $\lambda > 0$. Each customer's job has a value V which is independently and identically distributed with c.d.f. $F(x) = \Pr[V \leq x]$. In fact, the distribution function determines the demand structure of the users, and we call it the *demand distribution.* Customers submit their jobs if the job value is exceeding to the delay cost, or leave (balk) without submitting their jobs otherwise. The balking rule of a customer with job value V is formulated as follows.

$$\left\{ \begin{array}{lll} \text{join (submit jobs)} & \text{if} & V > c_n \ , \\ \text{leave} & \text{if} & V \leq c_n \ , \end{array} \right. \tag{2.1}$$

where $c_n = c\,(n+1)/\mu$, and n denotes the number of customers in system upon the customer's arrival. According to (2.1), the effective arrival rate for each n is $\lambda_n = \lambda \bar{F}(c_n),\ n = 0,1,2,\cdots$.

The net social welfare is defined as the difference of the total demand satisfied in the system and the delay cost, i.e., $m_s = m_v - m_c$ where $m_v \equiv \lambda \sum_{n=0}^{\infty} e_n \int_{c_n}^{\infty} x dF(x)$, $m_c \equiv \sum_{n=0}^{\infty} e_n \lambda_n c_n$ and $e_n,\ n = 0,1,2,\cdots$ is the steady state probability. It is shown that the net social welfare is written in a simple form.

Theorem 1

$$m_s = \lambda \sum_{n=0}^{\infty} e_n \int_{c_n}^{\infty} \bar{F}(x)dx. \tag{2.2}$$

Our primary interest is to investigate the relationships between the demand distribution F and the social welfare m_s under the existence of the externality. In the following sections, we derive sufficient conditions for monotonic relationships between the social welfare and demand distribution function.

3. Complete Monotone Demand Distribution It is hard to derive general statements on system behavior of the balking system with general demand distributions. However, if the demand is limited to a class of functions, one can see nice properties.

A distribution $F(x)$ is said to be complete monotone is there exists a distribution function $G(\eta)$ such that $\bar{F}(x) = \int_0^\infty e^{-\eta x} dG(\eta)$. The set of all complete monotone distribution functions is denoted by $\mathcal{CM}$.

Theorem 2 Suppose that $V_1 \sim \mathcal{CM}(G_1(\eta))$ and $V_2 \sim \mathcal{CM}(G_2(\eta))$. If either $G_1 \leq_{cx} G_2$, or $G_1 \geq_{st} G_2$, then $m_{s1} \leq m_{s2}$, where m_{sk} is the net social welfare under the demand structure V_k.

4. Constant Demand Distribution In this section, we investigate the balking system with homogeneous demand, i.e., constant job value. The system is the same one first introduced by Naor (1969), but the comparative analysis of the social welfare in the demand change is new.

Theorem 3 Consider the balking system with constant job values, and let $m_s(v)$ be the social welfare in the system with job value v and let $n_s(v)$ be the smallest integer that is not below $v\mu/c$. Suppose that $v_1 < v_2$, then it is shown that:

a) If $0 < n_s(v_1) = n_s(v_2)$, then $m_s(v_1) < m_s(v_2)$,

b) If $0 < \rho \leq 1$ and $n_s(v_1) + 2 \leq n_s(v_2)$, then $m_s(v_1) < m_s(v_2)$.

5. Role of Price Mechanism It is wel-known that in many simple balking systems, the price mechanism is effective for internalization of the congestion externality, and helps to increase the social welfare. We apply the price mechanism to systems with more complicated demand structure.

Let p be the process fee charged to users. Here, p is considered as an internal price, or a tariff controling the demand. Now, users' balking rule is modified as

$$\begin{cases} \text{join} & \text{if } V > c_n + p \ , \\ \text{leave} & \text{if } V \leq c_n + p \ . \end{cases} \tag{5.1}$$

On the other hand, the definition of the net social welfare stays the same. Now, one can decompose the social welfare into two components:

Theorem 4

$$m_s(p) = \lambda \sum_{n=0}^{\infty} e_n(p) \int_{c_n+p}^{\infty} \bar{F}(x)dx + p \sum_{n=0}^{\infty} e_n(p) \cdot \lambda_n(p) \tag{5.2}$$

Theorem 5 (Existence and Uniqueness of Optimal Price) Suppose that users have random job values V with c.d.f. $F(x) = 1 - e^{-\eta x}$, then there exists a unique maximizer $p = p^*$ for $m_s(p)$ that is stricly positive.

Theorem 6 Suppose that users in Group-1 and Group-2 have exponentially distributed job values V_1 and V_2 which have means $E[V_1] = 1/\eta_1$, $E[V_2] = 1/\eta_2$ respectively. Furthermore, suppose that Group-1 and Group-2 users face prices p_1 and p_2, respectively. We assume that $c/\mu > 1$ and $E[V_1]/p_1 = E[V_2]/p_2$. Then it is shown that if $E[V_1] < E[V_2]$, then $m_s(\eta_1, p_1) < m_s(\eta_2, p_2)$.

References Naor, P., 1969, On the regulation of queue size by levying tolls, Econometrica, Vol. 38, 13-24, etc.