Optimum Tree Networks for Single or Double Failure

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1 Introduction

We consider a problem to obtain a network of tree type which minimizes the probability that a requirement of communication is not realized when a single or double failure has occured.

Let $V = \{0, 1, \ldots, n-1\}$ be a set of n vertices and $\mathcal{T}$ be the whole set of undirected spanning trees on V. Assume that a nonnegative value r_{vu} (relative frequency of communication) is given to each pair of distinct vertices v and u in V, where $r_{vu} = r_{uv}$ holds. Also assume that, for a given value L (≥ 2),

$$\deg(v) \leq L \text{ holds for all } v \in V, \tag{1}$$

where $\deg(v)$ is the degree of vertex v. Let $\mathcal{E}_i$ denote the event that T has i failures, and let

$$\alpha_{ij} = \Pr\{i \text{ vertices and } j \text{ edges are broken down} \mid \mathcal{E}_{i+j}\}.$$

Also let $\mathcal{F}$ denote the event that a requirement of communication is not realized on T. By adding some more assumptions, we can show that minimizing $\Pr\{\mathcal{F} \mid \mathcal{E}_1\}$ is equivalent to minimizing

$$f_1(T) = \sum_{\{v,u\} \in \binom{V}{2}} d(v, u; T) r_{vu},$$

where $d(v, u; T)$ denotes the length of the path connecting v and u on T, and $\binom{V}{2}$ is the set of all pairs of distinct vertices in V. Similarly, minimizing $\Pr\{\mathcal{F} \mid \mathcal{E}_2\}$ is equivalent to minimizing

$$f_2(T) = \sum_{\{v,u\} \in \binom{V}{2}} (C + (2\alpha_{20} + 2\alpha_{11} - n) d(v, u; T)) d(v, u; T) r_{vu},$$

where $C = (n-2)(2n-3)\alpha_{20} + n(2n-3)\alpha_{02} + 2(n-1)(n-2)\alpha_{11}$. Our problem is to find a tree $T \in \mathcal{T}$ minimizing $f_i(T)$ $(i = 1, 2)$ under constraint (1). We say that a tree T minimizing f_i is *f_i-optimum* $(i = 1, 2)$.

As to the single-failure problem, Hu[4] showed that an f_1-optimum tree can be obtained by the Gomory-Hu algorithm[3] in the case *without* degree constraints; while Anazawa, Kodera and Jimbo[2] showed that a particular tree T^* is f_1-optimum subject to (1) under some conditions. Anazawa[1] gave a more general condition for the tree T^* to be f_1-optimum.

We will show in this paper that the tree T^* is also f_2-optimum under the same condition in [1].

2 Main Theorem and lemmas

Let $m_v = \sum_{u \neq v} r_{vu}$ for $v \in V$ and assume, without loss of generality, that $m_0 \geq m_1 \geq \cdots \geq m_{n-1}$ holds. We call a vertex v with $v \geq n$ a *dummy vertex*, and assume that $m_v = 0$ and $\deg(v) \leq L$ hold for all $v \geq n$.

The definition of the tree $T^* \in \mathcal{T}$ is as follows. For the maximum degree L, we set $s_0 = L$, $s_u = s_{u-1} + (L-1)$ for $u = 1, 2, \ldots$, and let N be the minimum integer satisfying $n - 1 \leq s_{N-1}$. Also we define a function π on a set $\{1, 2, \ldots, n-1\}$ by

$$\pi(v) = \begin{cases} 0 & \text{if } 1 \leq v \leq s_0 \\ u & \text{if } s_{u-1} + 1 \leq v \leq s_u \quad \text{for } u = 1, 2, \ldots, N-2 \\ N-1 & \text{if } s_{N-2} + 1 \leq v \leq n-1 \end{cases},$$

and let $E^* = \{e_1, e_2, \ldots, e_{n-1}\}$ be a set of edges, where $e_v = (\pi(v), v)$ for $v = 1, 2, \ldots, n-1$. Then T^* is defined by $T^* = (V, E^*)$. Figure 1 illustrates T^* for $n = 9$ and $L = 3$.

Main Theorem *A sufficient condition for T^* to be f_1- and f_2-optimum is as follows: if r_{vu}, $r_{vu'}$, $r_{v'u}$ and $r_{v'u'}$ are all defined for $v < v'$ and $u < u'$, then $r_{vu} - r_{vu'} \geq r_{v'u} - r_{v'u'}$ holds, where the equality sign holds if and only if both v and v' are dummy or both u and u' are dummy.*

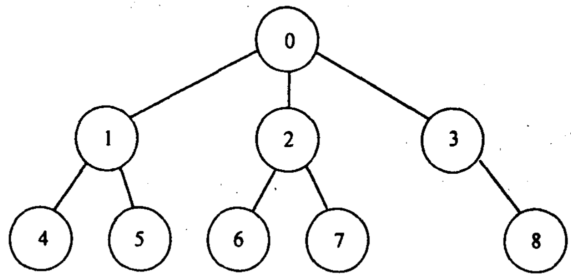

Figure 1: T^* for $n = 9$ and $L = 3$

Before proving Main Theorem, we present some lemmas without proofs. The first one is concerned with the tree T^*.

Lemma 1 *Let T^*_ν be a subtree of T^* consisting of $\{0, 1, \ldots, \nu-1\} \subset V$ ($\nu = 1, 2, \ldots, n$). For each T^*_ν ($\nu \geq 2$), let $P = (u_1, u_2, \ldots, u_k)$ be an arbitrary path of T^*_ν satisfying $u_1 < u_k$, and let $m = \lfloor \frac{k}{2} \rfloor$ where $\lfloor x \rfloor$ is the maximum integer not exceeding x. Then $u_i < u_{k-i+1}$ holds for $i = 1, 2, \ldots, m$.*

Next we show some lemmas concerned with f_1- or f_2-optimum trees satisfying the condition in Main Theorem. In the sequel of this paper, we use the following notation additionally. Let $P = (u_1, u_2, \ldots, u_k)$ be a path of a tree $T = (V, E) \in \mathcal{T}$, and a forest G is defined by $G = (V, E \setminus \{(u_1, u_2), (u_2, u_3), \ldots, (u_{k-1}, u_k)\})$. For the path $P = (u_1, u_2, \ldots, u_k)$, let $T(u_i) = (V(u_i), E(u_i))$ be the connected component containing u_i in G. Also, for a nonnegative integer l, let $V(u_i, l) = \{v \in V(u_i) | d(u_i, v; T) = l\}$.

Lemma 2 *Let T be an f_1- or f_2-optimum tree, and $P = (u_1, \ldots, u_k)$ be an arbitrary path of T such that $k = 2$ or 3 and $u_1 < u_k$ are satisfied. For any nonnegative integer l, if we can choose two vertices $v_1 \in V(u_1, l)$ and $v_2 \in V(u_k, l)$ arbitrarily, then $v_1 < v_2$ and $\deg(v_1) \geq \deg(v_2)$ hold.*

Remark The condition in Main Theorem is essencially utilized for the proof of Lemma 2.

Corollary 1 *Let $P = (u_1, u_2, \ldots, u_k)$ be a path of an f_1- or f_2-optimum tree T, and set $m = \lfloor \frac{k}{2} \rfloor$. If $u_m < u_{k-m+1}$ is satisfied, then $u_i < u_{k-i+1}$ holds for $i = 1, 2, \ldots, m$.*

Lemma 3 (Monotoneity) *Let T be an f_1- or f_2-optimum tree, and $P = (u_1, u_2, \ldots, u_k)$ be an arbitrary path of T satisfying $u_1 = 0$. Then $u_1 < u_2 < \cdots < u_k$ holds.*

Proof of Main Theorem (Outline) Let $T^* = (V, E^*) \in \mathcal{T}$ be the tree stated in Main Theorem, and $T = (V, E) \in \mathcal{T}$ be an f_1-optimum tree with $E \neq E^*$. Then we find that a certain path of T does not simultaneously satisfy Lemma 1, Corollary 1 and Monotoneity (Lemma 3). Hence, T^* must be f_1-optimum. In the same manner, we obtain the f_2-optimality of T^*. Therefore, T^* is f_1- and f_2-optimum.

The details of the proof will be shown in the session.

References

[1] T. Anazawa, On a Condition for Obtaining an Explicit Solution of Optimum Requirement Spanning Tree, *submitted to Networks.*

[2] T. Anazawa, T. Kodera and M. Jimbo, An Explicit Solution of Optimum Requirement Spanning Tree with Maximum Degree Conditions, *Congressus Numerantium*, **117** (1996) 43-52.

[3] R. E. Gomory and T. C. Hu, Multi-terminal network flows, *SIAM J. Appl. Math.*, **9** (1961) 551-570.

[4] T. C. Hu, Optimum Communication Spanning Trees, *SIAM J. Comput.*, **3** (1974) 188-195.