On Measuring the Inefficiency of DMU with Inner-Product Norm in DEA

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1 Introduction

Data Envelopment Analysis(DEA) is a useful non-parametric method to evalutate a relative efficiency of mult-input and mult-output units based on observed data. Its goals may be abbreviated by classifying the Decision Making Unit(DMU) into two groups; efficient or inefficient. However, the uncertainty like a measurement error should be incorporated in observed data, and clearly, it is important to assess the stability of efficiency classifications in DEA. In recent years a substantial amount of scholarly effort has been devoted to the development of efficiency measures considering the uncertainty ([1], [2] etc).

In this paper, the expansion of [2] for assessing the stability of classification is presented. An DMU's current activity serves as the center for a cell within which the DMU's classification remains unchanged under perturbation of the data. The maximal radius of such a cell may be interpreted as a measure of robustness for the classification under perturbations of the data (especially with respect to errors). We evaluete the maximal radius with the inner-product norm, while l_1, l_∞ (or Chebychev) norms are adopted in [2]. There are, at least, following three merits in adopting the ellipsoidal norm.

- It is a standard norm for a radius of the ball preserving DMU's classification.
- For inefficiet DMUs, the nearest efficient point is obtained, *i.e.*, we can find the most achievable improvement.
- Our formulation can be applied to the stocastic DEA model, which assumes that data may have normally distributed measurement error.

2 Formulation

We consider r DMUs. Let each activity of DMU$_j$ ($j = 1, \cdots, r$) be $\boldsymbol{z}_j^T \equiv (\boldsymbol{x}_j^T, \boldsymbol{y}_j^T) \in R^n$, where $\boldsymbol{x}_j \in R^m$ is a input vector, $\boldsymbol{y}_j \in R^s$ is a output vector and $n \equiv m + s$. Suppose that DMU$_1, \cdots$, DMU$_q$ ($q < r$) are classified into efficient DMUs in CCR-model, BCC-model or variants and that DMU$_0$ denotes the DMU being tested about the stability of its classification, where the subscript $0 \in \{1, \cdots, r\}$. Each DEA model assumes the theoretically possible input-output levels (*reference set* P) and it can be expressed by efficient activities $\boldsymbol{z}_1, \cdots, \boldsymbol{z}_q$. Also, let $P^{)0(}$ denote the reference set constructed excluding DMU$_0$'s activity.

The maximal radius problem for efficient DMU$_0$ is formulated as a convex quadratic programming problem:

$$\begin{array}{ll} \min & (\boldsymbol{z} - \boldsymbol{z}_0) \ \boldsymbol{Q}(\boldsymbol{z} - \boldsymbol{z}_0) \\ \text{s.t.} & \boldsymbol{z} \in P^{)0(} \end{array} \qquad (1)$$

, where $\boldsymbol{Q}$ is positive definite matrix

When the DMU$_0$ is inefficient, the maximal radius problem is reduced to the following nonconvex problem, whose feasible region is the complement of a convex set.

$$\begin{array}{ll} \min & f(\boldsymbol{z}) = (\boldsymbol{z} - \boldsymbol{z}_0) \ \boldsymbol{Q}(\boldsymbol{z} - \boldsymbol{z}_0) \\ \text{s.t.} & \boldsymbol{z} \in \overline{R^n \setminus P} \end{array} \qquad (2)$$

, where $\overline{X}$ denotes the closure of the set X. Problem(1) is a convex programming problem and is rather easy to solve by using some of typical algorithms. On the other hand, Problem(2) has a nonconvex body and is a difficult problem. Therefore, we primaly deal with this difficult problem, and will show that Problem(2) is reduced to the Linear Reverse Convex Programming (LRCP) problem and the Concave Quadratic Programming (CQP) problem.

3 Transformation

3.1 Linear Reverse Convex Program

At first, we give some following definitions. In BCC-model,

$e_i \in R^n$: the i unit vector

$$D = \begin{pmatrix} z_1 & \cdots & z & \tilde{I} \\ 1 & 1 & 1 & 0^T \end{pmatrix} \in R^{(n+1)\times(q+n)}$$

$$\tilde{I} \equiv (e_1, \cdots, e\ \ , -e_{+1}\ \ , \cdots, -e_{+}\ \ ) \in R^{n \times n}$$

$$x = \begin{pmatrix} y \\ x_0 \end{pmatrix} \in R^{n+1},\ \tilde{z}_0 = \begin{pmatrix} z_0 \\ 1 \end{pmatrix} \in R^{n+1}$$

$$U = \begin{pmatrix} Q^{-1} & 0 \\ 0^T & 0 \end{pmatrix} \in R^{(n+1)\times(n+1)}.$$

In CCR-model, the last row and column vectors of matrix U and the last element of vectors (x and $\tilde{z}_0$) are dropped, *i.e.*, $D = \left(z_1, \cdots, z\ , \tilde{I}\right) \in R^{n\times(q+n)}$, $x = y \in R^n$, $\tilde{z}_0 = z_0 \in R^n$, and $U = Q^{-1}$. Then, for inefficient DMU_0, consider the following problem including one nonconvex constraint :

$$\begin{array}{lll} \min & g(x) = -\tilde{z}_0^T x & \\ \text{subject to} & D^T x \leq 0 & \text{(LRCP)} \\ & x^T U x \geq 1. & \end{array}$$

Theorem 3.1 *Local and global minimizers of (LRCP) are one-to-one corresponding to those of problem(2).*

We can unify some DEA-models (CCR, BCC, and variants) when measuring the inefficiency of DMU_0.

3.2 Concave Quadratic Program

As well as the LRCP problem, we will show the equivalence between problem(2) and the concave quadratic program (CQP) proposed below. The definitions of D, x, $\tilde{z}_0$ and U are same as those of 3.1.

$$\begin{array}{lll} \max & g(x) = x^T U x & \\ \text{subject to} & D^T x \leq 0 & \text{(CQP)} \\ & -\tilde{z}_0^T x \leq 1 & \end{array}$$

Theorem 3.2 *Local and global minimizers of (CQP) are one-to-one corresponding to those of problem(2).*

4 Algorithm

For the concave minimization problem such as (CQP), enumerative algorithms based purely upon cutting planes are suggested. (See [3].) It is known that an optimal solution is always found after a finite number of iterations, if we combines *γ-valid cut* used in pure cutting plane approaches and *facial cuts* created by Majthay and Whinston ([4]). In these algorithm, simplex-based search for a local optimum is proposed, however, it takes much computation in exchanging basic and nonbasic variables, since the quadratic term of the objective function requires additional operations between matrices. Therefore, we propose the technique of the simplex-based search for (LRCP), which need not the exchange of basic and nonbasic variables for matrix U. Also, we arrange the construction of these two kind of cuts for (LRCP) and propose the finite cutting algorithm.

5 Simple Example

We test our technique with the simple example using the actual depertment-store data set in 1997. We will show some results in the presentation.

References

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