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Computational Aspects of Bilinear Matrix Inequality Problems

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Introduction

Many problems in control and system theory can be formulated using Linear Matrix Inequalities (LMIs) [1] which we can interpret as a Semidefinite Program (SDP) [7] in the field of optimization. The intensive research and advances in interior-point methods in these few years have provided polynomial complexity algorithms for SDPs [5] and have increased the importance of LMIs in control theory as well. The main focus of this talk is to propose an algorithm to solve a bilinear extension of the LMI known as the *Bilinear Matrix Inequality* (BMI). BMIs have an advantage of describing control systems more precisely than LMIs, though it is known that they are very difficult problems in practice [6]. Our research consists in analyzing the particular bilinear structure of BMIs by a Branch-and-Bound algorithm.

Consider then $k \times k$-symmetric matrices $\boldsymbol{B}_{ij}$ $(i = 0, 1, \cdots, m;\quad j = 0, 1, \cdots, n)$, and define a bilinear combination of them as:

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y}) &= \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j \boldsymbol{B}_{ij} + \sum_{i=1}^{m} x_i \boldsymbol{B}_{i0} \\ &\quad + \sum_{j=1}^{n} y_j \boldsymbol{B}_{0j} + \boldsymbol{B}_{00},\end{aligned}$$

$$(\boldsymbol{x}, \boldsymbol{y}) \in \Re^m \times \Re^n, \quad \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y}) \in \Re^{k \times k}$$

The Bilinear Matrix Inequality (BMI) is a constraint of the form:

$$\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y}) \preceq O \qquad (1)$$

i.e., a bilinear combination of the matrices $\boldsymbol{B}_{ij}$ which is a negative semidefinite matrix. Particularly, when $\boldsymbol{B}_{ij} = 0$ $(i = 1, 2, \cdots, m;\quad j = 1, 2, \cdots, n)$, (1) becomes an LMI.

We introduce now an optimization problem that we name the *BMI eigenvalue problem* (BMIEP) and includes (1) in its restriction.

$$\text{(BMIEP)} \quad \left\{ \begin{array}{ll} \min & \lambda \\ s.t. & \lambda \boldsymbol{I} - \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y}) \succeq 0 \\ & (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{F} \end{array} \right. \qquad (2)$$

where $\mathcal{F} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \Re^m \times \Re^n : \quad \underline{\boldsymbol{x}} \leq \boldsymbol{x} \leq \overline{\boldsymbol{x}}, \underline{\boldsymbol{y}} \leq \boldsymbol{y} \leq \overline{\boldsymbol{y}}\}$ is a box constraint.

Solving the optimization problem (2), we can determine if there is or not a point in $\mathcal{F}$ that satisfies (1). Answering this question is an important problem in robust control known as the *BMI feasible problem*. From the discussion given in the following section, we notice also that our branch-and-bound algorithm works for any linear objective function and even for any additional LMIs or BMIs in the restriction of (2).

A Branch-and-Bound Algorithm

At each iteration of our branch-and-bound algorithm, we solve a Linear Relaxed Problem of the minimization problem BMIEP. Renaming the bilinear terms $x_i y_j$ by w_{ij} in (2), we transform it in an SDP [2]. Further, we introduce some hyperplanes that restrict the domain of the corresponding linear minimization problem.

The *Linearized BMI eigenvalue Problem* (BMIEP_L) becomes:

$$(BMIEP_L) \quad \left\{ \begin{array}{ll} \min & \lambda \\ s.t. & \lambda \boldsymbol{I} - \boldsymbol{B}_L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}) \succeq 0 \\ & (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}) \in \tilde{\mathcal{F}}_w \end{array} \right. \qquad (3)$$

where

$$\begin{aligned}\boldsymbol{B}_L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}) &= \sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij} \boldsymbol{B}_{ij} + \sum_{i=1}^{m} x_i \boldsymbol{B}_{i0} \\ &\quad + \sum_{j=1}^{n} y_j \boldsymbol{B}_{0j} + \boldsymbol{B}_{00},\end{aligned}$$

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}) \in \Re^m \times \Re^n \times \Re^{mn}, \quad \boldsymbol{B}_L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}) \in \Re^{k \times k}$$

and

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$$\begin{aligned}\tilde{\mathcal{F}}_w \;=\; \{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{w}) \in \Re^m \times \Re^n \times \Re^{mn} :\; & \boldsymbol{w} \le \boldsymbol{x}\overline{\boldsymbol{y}} + \underline{\boldsymbol{x}}\boldsymbol{y} - \underline{\boldsymbol{x}}\overline{\boldsymbol{y}}, \\ & \boldsymbol{w} \le \boldsymbol{x}\underline{\boldsymbol{y}} + \overline{\boldsymbol{x}}\boldsymbol{y} - \overline{\boldsymbol{x}}\underline{\boldsymbol{y}}, \\ & \boldsymbol{w} \ge \boldsymbol{x}\underline{\boldsymbol{y}} + \underline{\boldsymbol{x}}\boldsymbol{y} - \underline{\boldsymbol{x}}\underline{\boldsymbol{y}}, \\ & \boldsymbol{w} \ge \boldsymbol{x}\overline{\boldsymbol{y}} + \overline{\boldsymbol{x}}\boldsymbol{y} - \overline{\boldsymbol{x}}\overline{\boldsymbol{y}}\}\end{aligned}$$

Proposition

(i) $\tilde{\mathcal{F}}_w \supseteq \mathcal{F}_w = \{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{w}) \in \Re^m \times \Re^n \times \Re^{mn} : (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{F}, x_i y_j = w_{ij}, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n\}$

(ii) The optimal value of the BMIEP_L (3) gives a lower bound of the optimal value of the BMIEP (2).

An upper bound for the BMIEP can be computed using the following heuristic. Once we fix $\boldsymbol{x}$ or $\boldsymbol{y}$, the restriction of (2) becomes an LMI. Therefore, minimizing alternatively the problem (2) with respect to $\boldsymbol{x}$ with $\boldsymbol{y}$ fixed, and vice versa, we obtain a reasonable approximation of a local optimal.

We propose a variation of the bisection method to generate each subproblem in our branch-and-bound algorithm. Information concerning about the solution of the BMIEP_L is used as a heuristic for our branching process.

We implemented our branch-and-bound algorithm in C++, and we solved several randomly generated problems. We incorporated the SDP solver SDPA [3] in its subroutines.

Concluding Remarks

The algorithm proposed in this talk is one of the first ones that seriously challenges to solve BMI problems. Our numerical experiments show that BMI problems with the dimension of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ rounding up to 7 are solved in an acceptable time for randomly generated problems. From the formulation we adopted, one can conclude that the size of the SDP solved in each subiteration of the branch-and-bound algorithm depends strongly on the dimension m and n of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, other than the dimension k of the matrices.

A further topic for improvements in this line of research is to analyze the convex hull of $\mathcal{F}_w$ that it is not well know for n and m simultaneously greater than 1. Also, we can work with SDP relaxations instead of Linear relaxation of the BMIs using the recently proposed Successive Convex Relaxation Method of Kojima and Tunçel [4].

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