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On the Geometry of an Optimal Maintenance Policy with Discounting

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1. INTRODUCTION

An attempt to solve the optimal maintenance problem on the graph was made first by Bergman [1]. That is, he proposed the graphical method on the scaled total time on test (TTT) curve to determine the optimal age replacement time which minimizes the expected cost per unit time in the steady-state. The merit of this method for the analytical approach has been considered to give a better insight for the corresponding statistical inference problem. In fact, Bergman and Klefsjö [2] and Dohi *et al.* [3] analyzed several non-parametric maintenance problems applying the scaled TTT concept. Recently, Dohi *et al.* [4] developed a new method based on the Lorenz curve to solve the different kind of maintenance problem.

However, it should be noted that the computational merit for the graphical method has not been known for the simple maintenance problems above. In other words, the graphical method is essentially equivalent to the analytical one if our interest is to calculate the optimal maintenance policy under the complete knowledge of statistical information. In this paper, through a repair-limit replacement problem with discounting, we show that the graphical method can characterize the optimal maintenance policy easily on the wider class of policy space than the analytical one. This implies that the graphical method is the unified approach to derive the optimal maintenance policy and can characterize it under weaker assumptions.

2. MODEL DESCRIPTION

Consider a single unit system, where each spare is provided only by an order after a lead time L (> 0) and each failed unit is repairable. The original unit begins operating at time 0. The lifetime for each unit obeys the general distribution function $F(t)$ with density $f(t)$ and finite $1/\lambda$ (> 0). When the unit has failed, the decision maker has to select repair or replacement. Suppose that the decision maker has a subjective probability distribution function on the repair-completion time $G(t)$ with density $g(t)$ and finite mean $1/\mu$ (> 0). If he or she estimates that the repair is completed up to the time limit $t_0 \in [0, \infty)$, then the repair is immediately started at the failure time. It is assumed that the unit once repaired is as good as new. However, if he or she estimates that the repair time is greater than t_0, then the failed unit is scrapped at the failure time, the spare unit is ordered immediately and delivered after the lead time L. The time required for replacement is negligible for convenience. The cycle repeats itself again and again.

Let us define a discount factor β (> 0). In addition, we define the Laplace transforms of densities $f(t)$ and $g(t)$ by $\mathcal{L}\{f(\beta)\}$ and $\mathcal{L}\{g(\beta)\}$ with β. Also, the function $G(\cdot)$ is assumed to have an inverse function, *i.e.* $G^{-1}(\cdot)$, and to be absolutely continuous and strictly increasing. Without any loss of generality, we assume $G(0) = F(0) = 0$ and $\lim_{t\to\infty} G(t) = \lim_{t\to\infty} F(t) = 1$. The cost components are the following;

k_r (> 0): repair cost per unit time

k_f (> 0): shortage cost per unit time

c (> 0): ordering cost per unit spare

Let us formulate the expected total discounted cost over an infinite time horizon. The expected total discounted cost for one cycle is

$$E_V(t_0) = \mathcal{L}\{f(\beta)\}\Big[(k_r + k_f)\int_0^{t_0} e^{-\beta t}\overline{G}(t)dt + \Big\{k_f\int_0^{L} e^{-\beta y}dy + ce^{-\beta L} - \frac{k_r + k_f}{\beta} \times (1 - e^{-\beta t_0})\Big\}\overline{G}(t_0)\Big]. \tag{1}$$

where, in general, $\overline{\psi}(\cdot) = 1 - \psi(\cdot)$. The present value of a unit cost during one cycle is

$$\delta(t_0) = \mathcal{L}\{f(\beta)\}\left[\int_0^{t_0} e^{-\beta t}dG(t) + e^{-\beta L}\overline{G}(t_0)\right]. \tag{2}$$

Since the expected total discounted cost over an infinite time horizon becomes $V(t_0) = \sum_{j=0}^{\infty} E_V(t_0)\delta^j(t_0) = E_V(t_0)/\overline{\delta}(t_0)$, the problem is to derive the optimal repair-time limit $t_0^* \in [0, \infty)$ which satisfies $V(t_0^*) = \min_{0 \le t_0 < \infty} V(t_0)$.

Lemma 2.1: The function $V(t_0)$ is strictly convex in t_0 if $(k_r + k_f)/\beta > V(t_0)$ for all $t_0 \in [0, \infty)$.

Theorem 2.2: Suppose that $(k_r + k_f)/\beta > V(t_0)$ for all $t_0 \in [0, \infty)$. Define the nonlinear function;

$$q(t_0) = \Big\{\frac{(k_r + k_f)(1 - e^{-\beta t_0})}{\beta} - k_f \int_0^L e^{-\beta y} dy - ce^{-\beta L}\Big\}\bar{\delta}(t_0) - (e^{-\beta L} - e^{-\beta t_0})E_V(t_0). \quad (3)$$

(i) If $q(0) < 0$ and $q(\infty) > 0$, then there exists a finite and unique optimal repair-time limit t_0^* $(0 < t_0^* < \infty)$ which satisfies $q(t_0^*) = 0$.

(ii) If $q(0) \ge 0$, then the function $V(t_0)$ is strictly increasing and the optimal repair-time limit is $t_0^* = 0$.

(iii) If $q(\infty) \le 0$, then the function $V(t_0)$ is strictly decreasing and the optimal repair-time limit is $t_0^* \to \infty$.

3. GEOMETRICAL APPROACH

Putting $p = G(t_0) \in [0, 1]$, define $\xi_\beta(p) = \psi_\beta(p)/\psi_\beta(1)$, where

$$\psi_\beta(p) = \int_0^{t_0} e^{-\beta t} dG(t). \quad (4)$$

Then we have the following useful result.

Theorem 3.1:

(i) If $c\exp\{-\beta L\} + k_r(\exp\{-\beta L\} - 1)/\beta > 0$, then the problem is reduced to

$$\max_{0 \le p \le 1} : \frac{\xi_\beta(p) + B_y}{p + B_x}, \quad (5)$$

where

$$B_x = \frac{(k_r + k_f)/\beta}{\mathcal{L}\{f(\beta)\}\alpha} - \frac{\alpha + (k_r + k_f)/\beta}{\alpha}. \quad (6)$$

$$B_y = \frac{(k_r + k_f)e^{-\beta L}(1 - \mathcal{L}\{f(\beta)\})/\beta - \alpha}{\mathcal{L}\{f(\beta)\}\mathcal{L}\{g(\beta)\}\alpha}. \quad (7)$$

$$\alpha = ce^{-\beta L} + \frac{k_r}{\beta}(e^{-\beta L} - 1). \quad (8)$$

(ii) If $c\exp\{-\beta L\} + k_r(\exp\{-\beta L\} - 1)/\beta < 0$, then

$$\min_{0 \le p \le 1} : \frac{\xi_\beta(p) + B_y}{p + B_x}. \quad (9)$$

(iii) If $c\exp\{-\beta L\} + k_r(\exp\{-\beta L\} - 1)/\beta = 0$, then

$$\min_{0 \le p \le 1} : \frac{\exp\{-\beta L\}}{\mathcal{L}\{g(\beta)\}} p - \xi_\beta(p). \quad (10)$$

Remark 3.2: From Lemma 2.1, the condition $(k_r + k_f)/\beta > V(t_0)$ for convexity is equivalent to the following inequalities;

$$J_1 < ce^{-\beta L} - k_r(1 - e^{-\beta L})/\beta < J_2, \quad (11)$$

where

$$\begin{aligned} J_1 &= \mathcal{L}\{f(\beta)\}\Big\{k_f \int_0^L e^{-\beta y} dy + ce^{-\beta L} + \frac{k_r + k_f}{\beta} \times (1 + e^{-\beta L})\Big\} - \frac{k_r + k_f}{\beta}(\mathcal{L}\{f(\beta)\} + 1) \\ &< 0, \end{aligned} \quad (12)$$

$$J_2 = \frac{k_r + k_f}{\beta}(1 - \mathcal{L}\{f(\beta)\}) > 0. \quad (13)$$

Remark 3.3: Define the empirical distribution of the repair time as follows.

$$G_{in}(x) \equiv \begin{cases} i/n & \text{for } x_i \le x \le x_{i+1}, \\ 1 & \text{for } x_n \le x, \end{cases} \quad (14)$$

where, $i = 0, 1, 2, \ldots, n - 1$. Then we can obtain a non-parametric estimator of the function $\xi_\beta(p)$;

$$\xi_{in} = \frac{1 - (n - i)e^{-\beta x_i}/n}{1 - \beta \sum_{j=1}^n (n - j + 1)(x_j - x_{j-1})e^{-\beta x_j}/n} - \frac{\beta \sum_{j=1}^i (n - j + 1)(x_j - x_{j-1})e^{-\beta x_j}/n}{1 - \beta \sum_{j=1}^n (n - j + 1)(x_j - x_{j-1})e^{-\beta x_j}/n}. \quad (15)$$

Hence, replacing $\xi_\beta(p)$ and p by ξ_{in} and i/n, respectively, in Theorem 3.1, an estimator of the optimal repair-time limit which minimizes the expected total discounted cost over an infinite time horizon can be calculated on the two dimensional space $\mathcal{R}^2$.

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