Augmenting a $(k-1)$-Vertex-Connected Multigraph to an ℓ-Edge-Connected and k-Vertex-Connected Multigraph

02004044 Kyoto University * ISHII Toshimasa
01403794 Kyoto University NAGAMOCHI Hiroshi
01001374 Kyoto University IBARAKI Toshihide

1 Introduction

Let $G = (V, E)$ stand for an undirected multigraph with a set V of *vertices* and a set E of *edges*. The connectivity augmentation problem has been extensively studied as an important problem in the network design problem.

The local edge-connectivity $\lambda_G(x, y)$ for two vertices $x, y \in V$ is defined to be the minimum size of a cut in G that separates x and y (i.e., x and y belong to different sides of X and $V - X$), or equivalently the maximum number of edge-disjoint path between x and y. The local vertex-connectivity $\kappa_G(x, y)$ for two vertices $x, y \in V$ is defined to be the number of internally-disjoint paths between x and y in G. For a given integer ℓ (resp., k), we call G *ℓ-edge-connected* (resp., *k-vertex-connected*) if $\lambda_G(x, y) \geq \ell$ (resp., $\kappa_G(x, y) \geq k$) holds for every $x, y \in V$.

In this paper, we consider the problem of augmenting the edge-connectivity and the vertex-connectivity of a given graph G simultaneously by adding the smallest number of new edges. For two given integers ℓ and k, we say that G is *(ℓ, k)-connected* if G is ℓ-edge-connected and k-vertex-connected. Given a multigraph $G = (V, E)$, and two integers ℓ and k, the *edge- and vertex-connectivity augmentation problem*, denoted by EVAP(ℓ, k), asks to augment G by adding the smallest number of new edges to G so that the resulting graph G' becomes (ℓ, k)-connected. Recently, the authors proved that EVAP$(\ell, 2)$ can be solved in $O((nm + n^2 \log n) \log n)$ time [2], and that EVAP$(\ell, 3)$ can be solved in polynomial time (in particular, $O(n^4)$ time if an input graph is 2-vertex-connected) for any fixed integer ℓ [3, 4]. In this paper, we show that if an input graph G is $(k-1)$-vertex-connected $(k \geq 4)$, then G can be made ℓ-edge-connected and k-vertex-connected by adding at most 2ℓ surplus edges over the optimum in polynomial time.

2 Definitions

For a subset $V' \subseteq V$ in G, $G - V'$ denotes the subgraph induced by $V - V'$. For an edge set F with $F \cap E = \emptyset$, we denote $G = (V, E \cup F)$ by $G + F$. An edge with end vertices u and v is denoted by (u, v). A *partition* $X_1, \cdots, X_t$ of vertex set V means a family of nonempty disjoint subsets of V whose union is V, and a *subpartition* of V means a partition of a subset of V. For two disjoint subsets of vertices X, $Y \subset V$, we denote by $E_G(X, Y)$ the set of edges, one of whose end vertices is in X and the other is in Y, and also denote $c_G(X, Y) = |E_G(X, Y)|$. A *cut* is defined as a subset X of V with $\emptyset \neq X \neq V$, and the *size* of a cut X is denoted by $c_G(X, V - X)$, which may also be written as $c_G(X)$. A cut with the minimum size is called a *minimum cut*, and its size, denoted by $\lambda(G)$, is called the *edge-connectivity* of G. For a subset X of V, $\{v \in V - X \mid (u, v) \in E$ for some $u \in X\}$ is called the *neighbor set* of X, denoted by $\Gamma_G(X)$. Let $p(G)$ denote the number of components in G. A *disconnecting set* of G is defined as a cut S of V such that $p(G - S) > p(G)$ holds and no $S' \subset S$ has this property. If G is connected and does not contain K_n, then a disconnecting set of the minimum size is called a *minimum disconnecting set*, and its size, denoted by $\kappa(G)$, is called the *vertex-connectivity* of G. On the other hand, we define $\kappa(G) = 0$ if G is not connected, and $\kappa(G) = n - 1$ if G is connected and contains the complete graph K_n. For a vertex set S in G, we call the components in $G - S$ *the S-components*, and denote the family of all S-components by $\mathcal{C}(G - S)$.

A cut $T \subset V$ is called *tight* if $\Gamma_G(T)$ is a minimum disconnecting set in G. A tight set D is called *minimal* if no proper subset D' of D is tight. We denote the maximum number of pairwise disjoint minimal tight sets by $t(G)$.

2.1 Edge-Splitting

Given a multigraph $G = (V, E)$, a designated vertex $s \in V$, vertices $u, v \in \Gamma_G(s)$ (possibly $u = v$) and a nonnegative integer $\delta \leq \min\{c_G(s, u), c_G(s, v)\}$, we construct graph $G' = (V, E')$ from G by deleting δ edges from $E_G(s, u)$ and $E_G(s, v)$, respectively, and adding new δ edges to $E_G(u, v)$. We say that G' is obtained from G by *splitting* (s, u) and (s, v) by size δ.

Given a multigraph $G = (V, E)$ and $s \in V$ with $\lambda_G(x, y) \geq \ell$ for all pairs $x, y \in V - s$, a pair $\{(s, u), (s, v)\}$ of two edges in $E_G(s)$ is called *λ-splittable*, if the multigraph G' resulting from splitting edges (s, u) and (s, v) satisfies $\lambda_{G'}(x, y) \geq \ell$ for all pairs $x, y \in V - s$. It is known in [5] that there is always a λ-splittable pair two edges incident to s, if $c_G(s)$ is even and $\ell \geq 2$ holds.

Given a multigraph $G = (V, E)$ and $s \in V$ with $|V| \geq$

$k+2$ and $\kappa_G(x,y) \geq k$ for all pairs $x,y \in V-s$, a pair $\{(s,u),(s,v)\}$ of two edges in $E_G(s)$ is called *κ-splittable*, if the multigraph G' resulting from splitting edges (s,u) and (s,v) satisfies $\kappa_{G'}(x,y) \geq k$ for all pairs $x,y \in V-s$. It is known in [1] that there is always a κ-splittable pair $\{(s,u),(s,v)\}$, if one of the following holds; (i) $\kappa(G-s) = k-1 \geq 1$ and $t(G-s) \geq \max\{2k-2, k+2\}$ hold and $G-s$ has an S-component T with $p((G-s)-S) \geq 3$ and $|\Gamma_G(s) \cap T| \geq 2$ for some disconnecting set S in $G-s$, (ii) if $\kappa(G-s) = k-1 \geq 1$ and $t(G-s) \geq k+2$ hold and $G-s$ has a disconnecting set S with $p((G-s)-S) = 2$.

3 An Algorithm for EVAP(ℓ, k)

We now present a polynomial time algorithm for EVAP(ℓ,k) for a $(k-1)$-vertex-connected input multigraph.

Let $\beta(G) \equiv \max\{p(G-S) \mid S$ is a disconnecting set in $G\}$. To make a graph G (ℓ,k)-connected, it is necessary to add at least $\ell - c_G(X)$ edges to $E_G(X, V-X)$ for each cut X, to add at least $k - |\Gamma_G(X)|$ edges to $E_G(X, V-X)$ for each cut X with $V-X-\Gamma_G(X) \neq \emptyset$, and to add at least $p(G-S)-1$ edges to connect components of $G-S$ for each disconnecting set S in G.

Lower Bound: $\gamma(G) \equiv \max\{\lceil \alpha(G)/2 \rceil, \beta(G)-1\}$, where

$$\alpha(G) = \max\left\{\sum_{i=1}^{p}(\ell - c_G(X_i)) + \sum_{i=p+1}^{q}(k - |\Gamma_G(X_i)|)\right\}$$

and the max is taken over all subpartitions $\{X_1, \cdots, X_p, X_{p+1}, \cdots, X_q\}$ of V such that $q \geq p \geq 0$ and $V - X_i - \Gamma_G(X_i) \neq \emptyset$, $i = p+1, \cdots, q$. □

The sketch of our algorithm for solving the EVAP(ℓ, k) for a $(k-1)$-vertex-connected multigraph, denoted by Algorithm EV-AUG, is given as follows.

Algorithm EV-AUG

Input: An undirected multigraph $G = (V,E)$ with $|V| \geq k+1$, $\kappa(G) = k-1$, and an integer $\ell \geq k \geq 4$.

Output: A set of new edges F with $|F| \leq opt(G) + 2\ell$ such that $G^* = G+F$ satisfies $\lambda(G^*) \geq \ell$ and $\kappa(G^*) \geq k$.

Step I. (Adding vertex s and associated edges): If $t(G) \leq 2\ell+1$ holds, then after adding a new vertex s, we can add a set F_1 of new edges between s and V so that $|F_1| \leq \alpha(G)$ and the resulting graph $G_1 = (V \cup \{s\}, E \cup F_1)$ satisfies $c_{G_1}(X) \geq \ell$ for all cuts $X \subset V$. After setting $G' := G_1$ and $\overline{G'} := G_1 - s$, go to Step III. If $t(G) \geq 2\ell+2$ holds, then after adding a new vertex s, we can add a set F_1 of new edges between s and V so that $|F_1| = \alpha(G)$ and the resulting graph $G_1 = (V \cup \{s\}, E \cup F_1)$ satisfies $c_{G_1}(X) \geq \ell$ for all cuts $X \subset V$, $|\Gamma_G(X)| \geq k$ for all cuts $X \subset V$ with $V - X - \Gamma_{G_1}(X) \neq \emptyset$. After setting $G' := G_1$ and $\overline{G'} := G_1 - s$, go to Step II.

Step II. (Edge-splitting): While $t(\overline{G'}) \geq 2\ell+2$ holds, repeat the following procedure.

If $\beta(\overline{G'}) - 1 < \lceil t(\overline{G'})/2 \rceil$, then we split a λ-splittable and κ-splittable pair (s,u) and (s,v) which decreases $t(\overline{G'})$ by at least one. Set $G' := G' - \{(s,u),(s,v)\} + \{(u,v)\}$, $\overline{G'} := G' - s$, and go to Step II.

If $\beta(\overline{G'}) - 1 \geq \lceil t(\overline{G'})/2 \rceil$, then execute the following procedure. If there is a λ-splittable and κ-splittable pair (s,u) and (s,v) which decreases $\beta(\overline{G'})$ by one, after at most one undoing an edge-splitting and at most one replacing one edge incident to s, then we split (s,u) and (s,v), set $G' := G' - \{(s,u),(s,v)\} + \{(u,v)\}$, $\overline{G'} := G' - s$, and go to Step II. Otherwise we can add more $\beta(G) - 1 - \lceil \alpha(G)/2 \rceil$ new edges to obtain a (ℓ,k)-connected graph, after splitting all remaining edges incident to s such that all pairs are λ-splittable (according to [5]). Output a set of all added edges to G as an optimal solution.

Step III. (Edge augmentation): Now $t(\overline{G'}) \leq 2\ell+1$ holds and G' satisfies $c_{G'}(X) \geq \ell$ for all cuts $X \subset V$. Then by [6], we can make G' k-vertex-connected by adding at most $t(\overline{G'}) - 1$ new edges F'. So we can obtain a (ℓ,k)-connected graph $G'' + F'$, where G'' denotes the resulting graph from splitting all remaining edges incident to s such that all pairs are λ-splittable. From $t(\overline{G'}) \leq 2\ell+1$, the number of all added edges to G is at most $\lceil \alpha(G)/2 \rceil + 2\ell$.

Theorem 3.1 *For a multigraph G with $\kappa(G) \geq k-1$, G can be made (ℓ,k)-connected by adding at most $\gamma(G) + 2\ell$ new edges in polynomial time, where $\gamma(G) = \max\{\lceil \alpha(G)/2 \rceil, \beta(G)\}$.* □

References

[1] J. Cherian and R. Thurimella, *Fast algorithms for k-shredders and k-node connectivity augmentation*, Proceedings 28th ACM Symposium on Theory of Computing, 1996, pp. 37–46.

[2] T. Ishii, H. Nagamochi, and T. Ibaraki, *Augmenting edge-connectivity and vertex-connectivity simultaneously*, LNCS, Springer-Verlag, 8th ISAAC 1997, pp. 102-111.

[3] T. Ishii, H. Nagamochi, and T. Ibaraki, *Optimal augmentation to make a graph k-edge-connected and triconnected*, Proc. of 9th Annual ACM-SIAM Symposium on Discrete Algorithms, 1998, pp. 280-289.

[4] T. Ishii, H. Nagamochi and T. Ibaraki, *k-edge and 3-vertex connectivity augmentation in an arbitrary multigraph*, Lecture Notes in Computer Science 1533, Springer-Verlag, 9th ISAAC, 1998, pp. 159–168.

[5] L. Lovász, Combinatorial Problems and Exercises, North-Holland, 1979.

[6] W. Mader, *Ecken vom Grad n in Minimalen n-fach zusammenhängenden Graphen*, Arch. Math. Vol.23, 1972, pp. 219–224.