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# Successive Convex Relaxation Method applied to Nonlinear Programs

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## 1 Introduction

We are interested in finding the global optimal value of the following nonlinear program:

$$\text{(NLP)} \left\{ \begin{array}{ll} \max & \boldsymbol{c}^T \boldsymbol{x} \\ s.t. & \boldsymbol{x} \in \mathcal{F} \end{array} \right.$$

where

$$\mathcal{F} = \left\{ \boldsymbol{x} \in I\!\!R^n : \begin{array}{ll} g_j(\boldsymbol{x}) \leq 0, & 1 \leq j \leq m_g \\ h_k(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}_k \boldsymbol{x} + \boldsymbol{p}_k^T \boldsymbol{x} + \mu_k = 0, & 1 \leq k \leq m_h \end{array} \right\},$$

$\boldsymbol{c} \in I\!\!R^n$, $\boldsymbol{g} : I\!\!R^n \to I\!\!R^{m_g}$, and $\boldsymbol{h} : I\!\!R^n \to I\!\!R^{m_h}$ is a quadratic (possible linear) vector function.

We assume that the feasible region of the (NLP) is bounded, and that the vector function $\boldsymbol{g}(\cdot)$ is sufficiently smooth on the feasible region, *e.g.*, twice continuously differentiable [1].

In general, solving this type of problems is extremely hard and developing an efficient and a general framework to solve it is a significant subject. Our approach is based on the Successive Convex Relaxation Method (SCRM), originally proposed for nonconvex quadratic optimization problems [2, 3], which we extended to nonlinear programs.

We will focus on the implementation of this method which employs some heuristic schemes of [4].

## 2 LP relaxation of (NLP)

Let denote by $\mathcal{S}^n$ the space of $n \times n-$symmetric matrices.

Since $g_j(\cdot) \in \mathcal{C}^2 \quad (j = 1, 2, \cdots, m_g)$ from our assumption, we can find $\boldsymbol{Q}_j \in \mathcal{S}^n$ such that

$$\ell_j(\boldsymbol{x}) = g_j(\boldsymbol{x}) - \boldsymbol{x}^T \boldsymbol{Q}_j \boldsymbol{x}$$

becomes convex in $\mathcal{F} \ni \boldsymbol{x}$. Then, we can rewrite the feasible region of (NLP) as

$$\mathcal{F} = \left\{ \boldsymbol{x} \in I\!\!R^n : \begin{array}{ll} \ell_j(\boldsymbol{x}) + \boldsymbol{x}^T \boldsymbol{Q}_j \boldsymbol{x} \leq 0, & 1 \leq j \leq m_g \\ \boldsymbol{x}^T \boldsymbol{P}_k \boldsymbol{x} + \boldsymbol{p}_k^T \boldsymbol{x} + \mu_k = 0, & 1 \leq k \leq m_h \end{array} \right\}.$$

The LP relaxation of $\mathcal{F}$ is defined by

$$\hat{\mathcal{F}}^L = \left\{ \boldsymbol{x} \in I\!\!R^n : \begin{array}{ll} \exists \boldsymbol{X} \in \mathcal{S}^n \text{ such that} & \\ \ell_j(\boldsymbol{x}) + \boldsymbol{Q}_j \bullet \boldsymbol{X} \leq 0, & 1 \leq j \leq m_g \\ \boldsymbol{P}_k \bullet \boldsymbol{X} + \boldsymbol{p}_k^T \boldsymbol{x} + \mu_k = 0, & 1 \leq k \leq m_h \end{array} \right\},$$

where $\bullet$ denotes the canonical inner-product in $\mathcal{S}^n$. Observe that $\hat{\mathcal{F}}^L$ is a convex set, since it is defined by inequalities of convex functions, and equality of linear functions.

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# 3 Successive Convex Relaxation Method (SCRM)

The algorithm starts with a convex set $C_0 \supseteq \mathcal{F}$, *e.g.*, box constraints, and successively solves nonlinear convex programs with a linear objective function, and a feasible region defined by $\hat{\mathcal{F}}^L$ and other quadratic or linear constraints involving $\boldsymbol{x}$ and $\boldsymbol{X}$. The details can be found in the refereed papers. Then, it can be shown that the algorithm generates a sequence of values $\zeta_0 \geq \zeta_1 \geq \cdots$ which are upper bounds of the global maximization value of the (NLP). The algorithm terminates after it obtains a close approximation of the global maximization value whenever it is known.

# 4 Concluding remarks

The SCRM is a powerful framework to solve nonlinear programs in general. However, much of its implementation regarding efficient heuristic procedures are not certain, yet. In this talk, we will improve some heuristic procedures proposed in [4] like diminishing the number of constraints or including tighter convex constraints in order to increase its efficiency. Numerical results using the nonlinear program solver NUOPT will be shown in the talk for benchmark nonlinear programs of small size.


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