2000年度日本オペレーションズ・リサーチ学会
秋季研究発表会

A note on Asymmetric Power Index for Voting Games

01605000 Univ. of Tokyo *MATSUI Tomomi
Univ. of Tokyo UEHARA Yoshikata

1. Introduction

In this paper, we propose a new power index for voting game which is useful in situations that not all coalitions are equally likely to be formed. For calculating our index, we do not need to construct the ideology profile space. Thus we can drop the assumption of the random appearance of bills required by Owen [1] and Shapley [2]. Our power index includes the ordinary Shapley-Shubic index and the Deegan-Packel index as special cases. We also develop axioms for the proposed index and use them to prove the uniqueness. We compare our index with the existing power indices by using the data of the House of Councilors in Japan.

2. Definition of the Power Index

For any pair of sets S and $\{i\}$, we denote the union $S \cup \{i\}$ and the difference $S \setminus \{i\}$ by $S+i$ and $S-i$. Let $N = \{1, 2, \ldots, n\}$ be the set of voters. A voting game is a pair $G = (N, \mathcal{W})$ satisfying that $N \in \mathcal{W} \subseteq 2^N$ and $\forall F \supseteq \forall E \in \mathcal{W}$, $F \in \mathcal{W}$. Each coalition in $\mathcal{W}$ is called a *winning coalition*. The family of minimal winning coalitions is denoted by $\mathcal{W}^{\min}$. If a coalition E is not winning, E is called *losing*.

Given a voting game $G = (N, \mathcal{W})$, a *profile* p of G is a function $p : \mathcal{W} \to [0,1]$ satisfying that $\sum_{E \in \mathcal{W}} p(E) = 1$. For any winning coalition E, a *simple profile* $p_E : \mathcal{W} \to \{0,1\}$ is the profile of G satisfying that $p_E(F) = 1$ if and only if $E = F$.

Given a voting game $G = (N, \mathcal{W})$ and a winning coalition $E \in \mathcal{W}$, $G[E]$ denotes the voting game $(N, \mathcal{W}[E])$ where $\mathcal{W}[E] = \{F \subseteq N : F \cap E \in \mathcal{W}\}$. We denote the Shapley-Shubic power index of the voting game $G[E]$ by $\phi[E] \in [0,1]^N$.

Now we define a new index. Given a voting game $G = (N, \mathcal{W})$ and a profile p of G, the η-index $\eta(G, p) \in [0,1]^N$ is defined by $\eta(G,p) = \sum_{E \in \mathcal{W}} p(E) \phi[E]$.

The η-index includes the Shapley-Shubic index and the Deegan-Packel index as a special case. When the given profile is the special simple profile p_N, then the η-index is equivalent to the ordinary Shapley-Shubic index. If the profile is defined by

$$p(E) = \begin{cases} 1/|\mathcal{W}^{\min}| & (E \in \mathcal{W}^{\min}), \\ 0 & (\text{ otherwise }). \end{cases}$$

then the η-index is equivalent to the Deegan-Packel index.

3. Axiomatic Characterization

In the following, we introduce axioms which implies the η-index.

Axiom 1: If a player $i \in E$ satisfies that $\forall F \subseteq N - i$, $v(F) = v(F+i)$, then $\eta(G,p)_i = 0$.

Axiom 2: Let p_E be a simple profile. If a pair of players $i, j \in E$ satisfies that $v(F+i) = v(F+j)$ for all $F \subseteq N \setminus \{i,j\}$, then $\eta(G, p_E)_i = \eta(G, p_E)_j$.

Axiom 3: For any pair of voting games

$G_1 = (N, \mathcal{W}_1)$, $G_2 = (N, \mathcal{W}_2)$, and a simple profile p_E satisfying $E \in \mathcal{W}_1 \cap \mathcal{W}_2$, $\eta(G_1, p_E) + \eta(G_2, p_E) = \eta(G_\wedge, p_E) + \eta(G_\vee, p_E)$ where $G_\wedge = (N, \mathcal{W}_1 \cap \mathcal{W}_2)$ and $G_\vee = (N, \mathcal{W}_1 \cup \mathcal{W}_2)$.

Axiom 4: The sum total of η-index values is equal to 1; i. e., $\sum_{i \in N} \eta(G, p_E)_i = 1$.

Axiom 5: For any voting game G and a pair of profiles p_1 and p_2 of G, η-index satisfy the property that $\eta(G, \lambda p_1 + (1-\lambda)p_2) = \lambda\eta(G, p_1) + (1-\lambda)\eta(G, p_2)$ for all $\lambda \in [0, 1]$.

Axiom 6: For any voting game $G = (N, \mathcal{W})$ and a coalition $E \in \mathcal{W}$. η-index satisfies $\eta(G, p_E) = \eta(G[E], p_E)$.

Theorem A power index is the η-index if and only if it satisfies Axioms $1, \ldots, 6$.

4. House of Councilors in Japan

In the paper [3], Ono and Muto considered the House of Councilors in Japan as a weighted majority game defined by the Table 1 with 6 players. They also used the profile in Table 2. which is the patterns of yea/nay combinations and their frequencies of the nonunanimous votes that occurred during the period of 1989–1992.

Table 1: The House of Councilors in Japan (1989–1992) .

Liberal Democratic Party (LDP)	109
Social Democratic Party of Japan (SDPJ)	74
Komeito (Komei)	21
Japan Communist Party (JCP)	14
Democratic Socialist Party (DSP)	10
Rengo	12
quota / total	127 / 240

Table 2: Patterns of yea/nay combinations (1989–1992).

LDP	SDJP	Komei	JCP	DSP	Rengo	value
Y	Y	Y	N	Y	Y	85
Y	N	N	N	N	N	18
Y	N	N	N	Y	N	9
Y	N	Y	N	Y	Y	6
N	Y	Y	Y	Y	Y	6
Y	N	Y	N	Y	N	5
Y	N	Y	Y	Y	Y	3
Y	Y	Y	N	Y	N	1

Table 3 shows power indices. S-O index (k) shows the Shapley-Owen index defined in a k-dimensional ideological space. O-M (k) is defined in a similar way. When we calculate η-index, we set the quota to 121.

Table 3: Power indices.

index	LDP	SDJP	Komei	JCP	DSP	Rengo
S-S	0.567	0.117	0.117	0.067	0.067	0.067
Bz	0.844	0.156	0.156	0.094	0.094	0.094
D-P	0.333	0.117	0.117	0.144	0.144	0.144
S-O (1)	0	0	0.5	0	0.5	0
S-O (2)	0.155	0.032	0.211	0.144	0.458	0.000
O-M (1)	0	0	0.932	0	0.068	0
O-M (2)	0.639	0	0.292	0	0.068	0
O-M (3)	0.639	0.135	0.180	0	0.045	0
O-M (4)	0.707	0.007	0.105	0	0.135	0.045
O-M (5)	0.511	0.166	0.202	0	0.049	0.072
η-index	0.550	0.117	0.145	0.064	0	0.125

4. Discussions

The main difference of our index from the nonsymmetric Shapley-Owen index and Ono and Muto's method is that we do not need to assume the existence of ideological space. We measure the ideological difference among parties by the patterns of yea/nay combinations. A bias of the distribution of patterns defines the ideological distances among parties. It means that we do not need to identify the absolute position of each party in the ideological space and so the obtained result based on the relative difference among parties.

Reference

[1] G. Owen, Political Games, *Naval Res. Log. Quarterly*, 18 (1971), 345–355.

[2] L. S. Shapley, A comparison of power indices and a nonsymmetric generalization, P-5872, The Rand Corporation.

[3] R. Ono and S. Muto, Party power in the house of councilors in Japan: an application of the nonsymmetric Shapley-Owen index, *JORSJ*, 40 (1997), 21–32.