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A Slacks-based Measure of Super Efficiency in DEA

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1. Introduction

In most models of Data Envelopment Analysis (DEA), the best performers have efficiency score unity, and, from experience, we know that usually there are plural Decision Making Units (DMU) which have this "efficient status." To discriminate between these efficient DMUs is an interesting research subject. Several authors have proposed methods for ranking the best performers. We will call this problem the "super-efficiency problem."

In this paper, we discuss the "super-efficiency" issues based on the slack-based measure of efficiency (SBM).

2. Slacks-based Measure of Efficiency

We will deal with n DMUs (Decision Making Units) with the input and output matrices $X = (x_{ij}) \in R^{m\times n}$ and $Y = (y_{ij}) \in R^{s\times n}$, respectively. We assume that the data set is positive, i.e. $X > O$ and $Y > O$.

We consider an expression for describing a certain DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ as

$$\boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^- \quad (1)$$

$$\boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+, \quad (2)$$

with $\boldsymbol{\lambda} \geq \mathbf{0}$, $\boldsymbol{s}^- \geq \mathbf{0}$ and $\boldsymbol{s}^+ \geq \mathbf{0}$. We have:

$$\boldsymbol{x}_o \geq \boldsymbol{s}^-. \quad (3)$$

Using $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$, we define an index ρ as follows:

$$\rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{i=1}^{s} s_i^+/y_{io}}. \quad (4)$$

In an effort to estimate the efficiency of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, we formulate the following fractional program [SBM] in $\boldsymbol{\lambda}$, $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$.

$$\text{[SBM]} \quad \min\ \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{i=1}^{s} s_i^+/y_{io}} \quad (5)$$

$$\text{subject to } \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^-$$

$$\boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+$$

$$\boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{s}^- \geq \mathbf{0},\ \boldsymbol{s}^+ \geq \mathbf{0}.$$

Let an optimal solution for [SBM] be $(\rho^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$.

Definition 1 (SBM-efficient)
A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-efficient, if $\rho^* = 1$.

3. Super-efficiency evaluated by SBM

In this section, we discuss the super-efficiency issues under the assumption that the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-efficient, i.e. $\rho^* = 1$. Let us define a production possibility set $P \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$ spanned by (X, Y) excluding $(\boldsymbol{x}_o, \boldsymbol{y}_o)$. Further, we define a subset $\bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$ of $P \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$ as

$$\bar{P}\setminus(\boldsymbol{x}_o, \boldsymbol{y}_o) = P\setminus(\boldsymbol{x}_o, \boldsymbol{y}_o) \bigcap \{\bar{\boldsymbol{x}} \geq \boldsymbol{x}_o \text{ and } \bar{\boldsymbol{y}} \leq \boldsymbol{y}_o\}. \quad (6)$$

By the assumption $X > 0$ and $Y > 0$, $\bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$ is not empty.

As a weighted l_1 distance from $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ and $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$, we employ the index δ as defined by

$$\delta = \frac{\frac{1}{m}\sum_{i=1}^{m} \bar{x}_i/x_{io}}{\frac{1}{s}\sum_{r=1}^{s} \bar{y}_r/y_{ro}}. \quad (7)$$

From (6), this distance is not less than 1 and attains 1 if and only if $(\boldsymbol{x}_o, \boldsymbol{y}_o) \in \bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$, i.e. exclusion of the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ has no effect on the original production possibility set P.

We can interpret this index as follows. The numerator is a weighted l_1 distance from $\boldsymbol{x}_o$ to $\bar{\boldsymbol{x}}(\geq \boldsymbol{x}_o)$, and hence it expresses an average expansion rate of $\boldsymbol{x}_o$ to $\bar{\boldsymbol{x}}$ of the point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \bar{P}\setminus(\boldsymbol{x}_o, \boldsymbol{y}_o)$. The denominator is a weighted l_1 distance from $\boldsymbol{y}_o$ to $\bar{\boldsymbol{y}}(\leq \boldsymbol{y}_o)$, and hence it is an average reduction rate of $\boldsymbol{y}_o$ to $\bar{\boldsymbol{y}}$ of $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$. The smaller the denominator is, the farther $\boldsymbol{y}_o$ is positioned to $\bar{\boldsymbol{y}}$. Its inverse can be interpreted as an index of the distance from $\boldsymbol{y}_o$ to $\bar{\boldsymbol{y}}$. Therefore, δ is a product of two indices: one, the distance in the input space, and the other, that in the output space. Both indices are dimensionless.

Based on the above observations, we define the super-efficiency of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ as the optimal objective function value δ^* of the following program:

$$\text{[SuperSBM]} \quad \delta^* = \min \delta = \frac{\frac{1}{m}\sum_{i=1}^{m} \bar{x}_i/x_{io}}{\frac{1}{s}\sum_{r=1}^{s} \bar{y}_r/y_{ro}} \quad (8)$$

$$\text{subject to} \quad \bar{\boldsymbol{x}} \geq \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{x}_j$$

$$\bar{\boldsymbol{y}} \leq \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{y}_j$$
$$\bar{\boldsymbol{x}} \geq \boldsymbol{x}_o \text{ and } \bar{\boldsymbol{y}} \leq \boldsymbol{y}_o$$
$$\bar{\boldsymbol{y}} \geq \boldsymbol{0}, \ \ \boldsymbol{\lambda} \geq \boldsymbol{0}.$$

We have the following two propositions.

Proposition 1 *The super-efficiency score δ^* is units invariant, i.e. it is independent of the units in which the inputs and outputs are measured provided these units are the same for every DMU.*

Proposition 2 *Let $(\alpha \boldsymbol{x}_o, \beta \boldsymbol{y}_o)$ with $\alpha \leq 1$ and $\beta \geq 1$ be a DMU with reduced inputs and enlarged outputs than $(\boldsymbol{x}_o, \boldsymbol{y}_o)$. Then, the super-efficiency score of $(\alpha \boldsymbol{x}_o, \beta \boldsymbol{y}_o)$ is not less than that of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$.*

4. Input-Oriented Super-efficiency

In order to adapt our super-efficiency model to input orientation, we can modify the preceding program as follows.

For input orientation, we deal with the weighted l_1-distance only in the input space, keeping the outputs status quo. Thus, the program turns out to be:

$$[\text{SuperSBM(I)}] \qquad \delta^* = \min \delta = \frac{1}{m}\sum_{i=1}^{m} \bar{x}_i / x_{io} \quad (9)$$
$$\text{subject to} \qquad \bar{\boldsymbol{x}} \geq \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{x}_j$$
$$\bar{\boldsymbol{y}} \leq \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{y}_j$$
$$\bar{\boldsymbol{x}} \geq \boldsymbol{x}_o \text{ and } \bar{\boldsymbol{y}} = \boldsymbol{y}_o$$
$$\boldsymbol{\lambda} \geq \boldsymbol{0}.$$

The following proposition holds for this program:

Proposition 3 *If inputs $\boldsymbol{x}_o$ decrease to $\boldsymbol{x}_o - \Delta \boldsymbol{x}$ ($\geq \boldsymbol{0}$, $\Delta \boldsymbol{x} \geq \boldsymbol{0}$), then the optimal objective function value $\delta_I^*(\Delta \boldsymbol{x})$ corresponding to this change satisfies*

$$\delta_I^*(\Delta \boldsymbol{x}) \geq \delta_I^*. \quad (10)$$

Furthermore, the equality holds if and only if $\Delta x_i = 0$ or $\bar{x}_i^* = x_{io} - \Delta x_i$ holds for every i $(= 1, \ldots, m)$, where $\bar{x}_i^*$ is an optimal solution of the above program (9).

5. The Andersen and Petersen Model

Andersen and Petersen (1993) proposed the following super-efficiency model:

$$[\text{SuperCCR}] \qquad \theta^* = \min \theta \quad (11)$$
$$\text{subject to} \qquad \theta \boldsymbol{x}_o = \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{x}_j + \boldsymbol{s}^-$$
$$\boldsymbol{y}_o = \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{y}_j - \boldsymbol{s}^+$$
$$\boldsymbol{\lambda} \geq \boldsymbol{0}, \ \boldsymbol{s}^- \geq \boldsymbol{0}, \ \boldsymbol{s}^+ \geq \boldsymbol{0}.$$

Let an optimal solution of [SuperCCR] be $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$. For an efficient DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, θ^* is not less than unity, and this value indicates "super-efficiency." Regarding this measure we have the following proposition:

Proposition 4 *The* [SuperCCR] *model returns the same super-efficiency score θ^* for any DMUs represented by $(\boldsymbol{x}_o - \alpha \boldsymbol{s}^{-*}/\theta^*, \ \boldsymbol{y}_o)$ for the range $0 \leq \alpha \leq 1$.*

This contradicts our common understanding that a reduction of input values usually increases super-efficiency. This irrationality is caused by the fact that this model deals only with the radial measure and neglects the existence of input slacks as represented by $\boldsymbol{s}^{-*}$.

Furthermore, we have the following relationships between [SuperCCR] and [SuperSBM(I)].

Lemma 1 *Let us define*

$$\alpha^* = \min_i \left\{ \frac{(\theta^* - 1)x_{io}}{s_i^{-*}} \mid s_i^{-*} > 0 \right\} \quad (12)$$
$$= 0 \ \ \text{if} \ \ \boldsymbol{s}^{-*} = \boldsymbol{0}.$$

Then, $(\tilde{\boldsymbol{x}} = \theta^* \boldsymbol{x}_o - \alpha^* \boldsymbol{s}^{-*}, \tilde{\boldsymbol{y}} = \boldsymbol{y}_o, \tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^*)$ is a feasible solution for [SuperSBM(I)].

Let an optimal solution of [SuperCCR] be $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$, the optimal objective value of [SuperSBM(I)] be δ_I^*, and α^* as defined by (12). Then we have:

Theorem 1

$$\delta_I^* \leq \theta^* - \frac{\alpha^*}{m}\sum_{i=1}^{m} \frac{s_i^{-*}}{x_{io}}. \quad (13)$$

References

[1] Andersen, P. and N.C. Petersen, "A Procedure for Ranking Efficient Units in Data Envelopment Analysis," *Management Science*, 39, 1261-1264, 1993.

[2] Tone, K., "A Slacks-based Measure of Efficiency in Data Envelopment Analysis," *European Journal of Operational Research*, 130, 498-509, 2001.