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Optimal Replacement of a System According to a Semi-Markov Decision Process in a Semi-Markov Environment

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Abstract

This paper investigates an optimal replacement problem of a system in a semi-Markov environment. The system itself deteriorates according to a semi-Markov process, and is further influenced by its environment, which changes according to a semi-Markov process. Each change of the environment's state will change the parameters modelling the system and also cause damage on the system. For minimizing the discounted total costs, we show the existence of an optimal control limit policy. A special case of Markov environment is discussed to obtain better results, and the state space is reduced to be finite.

1 Introduction

Optimal replacement problem is an interesting research area for a long time (see, e.g., in Cho and Parlar (1991), Feldman (1976), Thangaraj and Stanly (1992), Yeh (1996)). It considers a system that will deteriorate and thus should be replaced by a new one when it deteriorated too bad. There are two types of the deterioration considered in reliability literature, one is due to the operation of the system while the other is caused by influence of the environment to the system. The influence occurs intermittently and is often described by a Poisson process. But in general these two types of deterioration are considered separately in literature.

This paper will further discuss it, but the system itself deteriorates according to a semi-Markov process. Also, the environment changes according to a semi-Markov process. Each change of the environment's state will change the parameters modelling the system and also cause damage on the system. We formulate it as a semi-Markov decision process in a semi-Markov environment, and show the existence of the optimal control limit policy for both finite and infinite horizon. It is simplified when the environment is Markov.

2 System Model

The system considered here is as follows:

(1) The system is in a semi-Markov environment $\{(J_n, L_n), n \geq 0\}$ with a kernel $G_{kk'}(t)$ on a set K of countable environment states. Let $\psi_{kk'} = G_{kk'}(\infty)$ and $G_k(t) = \sum_{k'} G_{kk'}(t)$.

(2) During an environment state k, the system itself operates according to a semi-Markov process with the kernel $\{P_{ij}^k(t), i, j \in S\}$ and countable state set $S = \{0, 1, 2, \ldots\}$, where the state 0 represents a new system, and states $1, 2, \ldots$ represent the different degrees of deterioration of the system, and the bigger the value, the more serious the deterioration. Let $p_{ij}^k = P_{ij}^k(\infty), T_{ij}^k(t) = P_{ij}^k(t)/p_{ij}^k$, and $T_i^k(t) = \sum_j P_{ij}^k(t)$.

(3) Suppose that the environment is in state k, then one of the following two actions can be chosen if the system state transfers to i:

- operate continually the system (denoted by O) with a cost rate $h^k(i)$;
- replace the system by a new one (denoted by R) with a cost rate $c^k(i)$, and the time of the replacement is assumed to be a random variable with distribution function $F(t)$, and the state after replacement will be 0.

(4) When the environment state changes from k to k', if action O is being chosen then the system state will change immediately according to a probability q_{ij}^k and an instantaneous cost $R^k(i, 0)$ occurs; while if action R is chosen then the replacement is immediately completed and an instantaneous cost $R^k(i, R)$ occurs.

(5) The objective is to minimize the expected discounted total costs with discount factor $\alpha > 0$.

Such a system is modelled by a semi-Markov decision processes (SMDP) in a semi-Markov environment.

3 Optimal Control Limit Policy

We introduce the following assumption.
Assumption For each $k \in K$,

A.1) $\sum_{j \geq m} q_{ij}^k$ is nondecreasing in i, $\forall m \geq 0$;

A.2) $h^k(i), c^k(i), R^k(i,O)$ and $R^k(i,R)$ are all nonnegative and nondecreasing in i;

A.3) both $h^k(i) - c^k(i)$ and $R^k(i,O) - R^k(i,R)$ are nondecreasing in i;

A.4) $F(t) \geq T_0^k(t) \geq T_1^k(t) \geq T_2^k(t) \geq T_3^k(t) \geq \cdots$, $\forall t \geq 0$, i.e., $T_i^k(\cdot)$ is stochastically nondecreasing in i and $F(\cdot)$ is the smallest.

A.5) $\int_0^\infty e^{-\alpha t} \sum_j V(t,j) p_{ij}^k T_{ij}^k(dt)$ is nondecreasing in i for each function V if $V(t,j)$ is nonnegative and nondecreasing in i for each $t \geq 0$.

As usual, $\sum_{j \geq m} q_{ij}^k$ is nondecreasing in i means that the bigger the deteriorate degree of the system, the faster of deterioration resulted by the environment state change. $h^k(i) - c^k(i)$ is nondecreasing in i indicates that the operation cost increases faster than replacement cost as the increasing of deteriorate degree of the system, and similar for $R^k(i,O) - R^k(i,R)$.

Let $x = (k,s,i) \in \Omega = \{(k,s,i) : k \geq 0, s \geq 0, i \in S\}$ be the mathematical state which means that the environment is in state k since time s ago and the system's state just transferees to i.

For $n \geq 0$, let $V_n^*(x)$ be the optimal value from state x in n stages, while $V_n^*(x,a)$ be the value from state x in n stages if action a is taken in the first stage and then the optimal policy in the remaining stages. Denote by

$$v_n(x) = V_n^*(x,O) - V_n^*(x,R)$$

Similarly, let $V^*(x), V^*(x,a)$ for the infinite horizons, and $v(x) = V^*(x,O) - V^*(x,R)$.

Then we can obtain the following theorem.
Theorem 3.1 *Under Assumption A, both $V_n^*(k,s,i)$ and $v_n(k,s,i)$ are nondecreasing in i for each $n \geq 0, k \in K, s \geq 0$, so*

$$V_n^*(k,s,i) = \begin{cases} V_n^*(k,s,i,O), & 0 \leq i < i_n^*(k,s) \\ V_n^*(k,s,i,R), & i \geq i_n^*(k,s) \end{cases}$$

where

$$i_n^*(k,s) := \min\{i | v_n(k,s,i) \geq 0\}$$

Similarly, both $V^*(k, s, i)$ and $v(k, s, i)$ are also nondecreasing in i, and

$$V^*(k,s,i) = \begin{cases} V^*(k,s,i,O), & 0 \leq i < i^*(k,s) \\ V^*(k,s,i,R), & i \geq i^*(k,s) \end{cases}$$

where

$$i^*(k,s) := \min\{i | v(k,s,i) \geq 0\}$$

4 Markov Environment

In this section, we consider that the environment is Markov: for $k, k' \in K$,

$$G_{kk'}(t) = \psi_{kk'} G_k(t),\ G_k(t) = 1 - e^{-\lambda_k t},\ t \geq 0$$

In this case, it will be shown that the variable "s" in state $x = (k,s,i)$ can be delimited.
Theorem 4.1 *Under Assumption A.1), A.2), A.3), A.4) and A.5), for $k \in K$ and $n \geq 1, v_n(k,i) := V_n^*(k,i,O) - V_n^*(k,i,R)$ is nondecreasing in i, so $v_n(k,i) < 0$ iff $i < i_n^*(k) := \min\{i : v_n(k,i) \geq 0\}$; moreover, $v(k,i) := V^*(k,i,O) - V^*(k,i,R)$ is also nondecreasing in i, and $v(k,i) < 0$ iff $i < i^*(k) := \min\{i : v(k,i) \geq 0\}$. Thus, there exist optimal control limit policies.*

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