A paradox of concurrent convergence method for a typical mutual evaluation system

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1. Introduction

Saaty [3] extends a hierarchy structure of criteria and alternatives into a network one, and proposes Analytic Network Process (ANP). In ANP the relative weights of criteria and alternatives is simultaneously provided from evaluation values between criteria and alternatives. However, it is hard for a decision maker to evaluate criteria from alternatives uniquely and hence, evaluation values of criteria from alternatives are often very unstable or erroneous. In order to overcome this difficulty, Kinoshita and Nakanishi [2] propose the Concurrent Convergence Method(CCM) which determines not only unique evaluation values but also the relative weights of alternatives simultaneously. Introducing a definition of relative weights of criteria by CCM, this study presents that CCM provides relative weights of criteria whose rank disagrees with some consensus one of criteria of alternatives.

2. Overall weights for criteria in CCM

Firstly we introduce CCM [2] briefly. Let I and J be a set of alternatives and that of criteria, respectively, then we denote the evaluation value of alternative $i \in I$ from criteria $j \in J$ by a_{ij}. Let K be a set of a regulating alternative which plays a role of a yardstick in CCM evaluation process. In CCM a decision maker evaluates alternatives relative to each regulating alternative $k \in K$ under all criteria and by one step of CCM we have an evaluation matrix A whose (i,j) component is a_{ij}. Let A_k be a diagonal matrix whose (j,j) component is a_{kj}, then AA_k^{-1} is regarded as an evaluation matrix when the regulating alternative k is a yardstick in the evaluation process.

In CCM the decision maker evaluates criteria from the viewpoint of each regulating alternative $k \in K$ and then we have a evaluation vector $\boldsymbol{b}^k$ of criteria from each regulating alternative $k \in K$. A main process of CCM changes $\{\boldsymbol{b}^i | i \in K\}$ into $\{\hat{\boldsymbol{b}}^i | i \in K\}$ such that

$$\frac{A_k^{-1}\hat{\boldsymbol{b}}^k}{\boldsymbol{e}^\top A_k^{-1}\hat{\boldsymbol{b}}^k} = \frac{A_l^{-1}\hat{\boldsymbol{b}}^l}{\boldsymbol{e}^\top A_l^{-1}\hat{\boldsymbol{b}}^l} \tag{2.1}$$

for all $k, l \in K$. From (2.1) we have

$$\frac{AA_k^{-1}\hat{\boldsymbol{b}}^k}{\boldsymbol{e}^\top AA_k^{-1}\hat{\boldsymbol{b}}^k} = \frac{AA_l^{-1}\hat{\boldsymbol{b}}^l}{\boldsymbol{e}^\top AA_l^{-1}\hat{\boldsymbol{b}}^l} \tag{2.2}$$

for all $k, l \in K$, where $\boldsymbol{e}$ is all one vector. Kinoshita and Nakanishi focus the following normal vector generated from the view-point of only regulating alternative k:

$$\frac{AA_k^{-1}\hat{\boldsymbol{b}}^k}{\boldsymbol{e}^\top AA_k^{-1}\hat{\boldsymbol{b}}^k}. \tag{2.3}$$

For (2.3) we see that

1. AA_k-1 is the evaluation matrix of alternatives relative to regulating alternative k,
2. $\hat{\boldsymbol{b}}^k$ is the adjusted evaluation vector of criteria from the view-point of regulating alternative k,
3. $\boldsymbol{e}^\top AA_k^{-1}\hat{\boldsymbol{b}}^k$ is the normalizing discount factor of $AA_k^{-1}\hat{\boldsymbol{b}}^k$.

Since it follows from (2.2) that (2.3) is constant with the choice of $k \in K$, it can be defined as the overall weight vector of an alternative, which is denoted by $\boldsymbol{p}$.

From 1. A_k is the discount factor of A for regulating alternative k. Therefore, for an average alternative $\frac{1}{|I|}\sum_{i\in I} A_i$ can be regarded as a discount factor of A. In fact, since $\sum_{i\in I} A_i$ is a diagonal matrix whose (j,j) component is a sum of the j^{th} column of A, $A\left(\sum_{i\in I} A_i\right)^{-1}$ has all column sums equal to 1, which is a typical column-wise standardizing evaluation matrix in ANP. Therefore, we define $\boldsymbol{q}$ satisfying

$$A(\textstyle\sum_{i\in I} A_i)^{-1}\boldsymbol{q} = \boldsymbol{p} \quad \text{and} \tag{2.4}$$

$$\boldsymbol{e}^\top \boldsymbol{q} = 1 \tag{2.5}$$

as the overall weight vector of criteria. The following theorems guarantee the existence and uniqueness of the overall weight vector $\boldsymbol{q}$:

Theorem 1 *Let $\boldsymbol{r}$ be $A_k^{-1}\hat{\boldsymbol{b}}^k/(\boldsymbol{e}^\top A_k^{-1}\hat{\boldsymbol{b}}^k)$ satisfying (2.1) for some $k \in K$, then the overall weight vector $\boldsymbol{q}$ of criteria is $(\sum_{i\in I} A_i)\,\boldsymbol{r}/(\boldsymbol{e}^\top(\sum_{i\in I} A_i)\,\boldsymbol{r})$.*

Theorem 2 *Let $I = \{1,\dots,m\}$ and suppose $I = K$, then a positive principal eigenvector of*

$$\begin{bmatrix} \hat{\boldsymbol{b}}^1 & \cdots & \hat{\boldsymbol{b}}^m \end{bmatrix} A(\textstyle\sum_{i\in I} A_i)^{-1} \tag{2.6}$$

has the same direction of $\boldsymbol{q}$. Hence, let $[\boldsymbol{x}^\top, \boldsymbol{y}^\top]^\top$ be a positive principal eigenvalue of a supermatrix

$$\begin{bmatrix} 0 & \hat{\boldsymbol{b}}^1 \cdots \hat{\boldsymbol{b}}^m \\ A(\sum_{i\in I} A_i)^{-1} & 0 \end{bmatrix}, \tag{2.7}$$

then $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *have the same direction of* $\boldsymbol{q}$ *and* $A(\sum_{i\in I} A_i)^{-1}\boldsymbol{q}$*, respectively.*

Takahashi [5] defines $\boldsymbol{x}$ of (2.7) as the overall weight vector of criteria and hence, his definition coincides with $\boldsymbol{q}$. From some numerical examples Kinoshita [1] predicts $\boldsymbol{y} = \boldsymbol{p}$ without a mathematical proof.

3. A numerical example of a paradox

We show a numerical example with 2 criteria and 3 alternatives. Suppose that $I = K = \{1, 2, 3\}$, $J = \{1, 2\}$,

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1/2 \\ 3 & 1/6 \end{bmatrix}, \boldsymbol{b}^1 = \begin{bmatrix} 0.56 \\ 0.44 \end{bmatrix}, \boldsymbol{b}^2 = \begin{bmatrix} 0.53 \\ 0.47 \end{bmatrix} \quad (3.1)$$

and

$$\boldsymbol{b}^3 = \begin{bmatrix} 0.51 \\ 0.49 \end{bmatrix}. \quad (3.2)$$

From $\boldsymbol{b}^1, \boldsymbol{b}^2$ and $\boldsymbol{b}^3$ we see that all alternatives prefer criterion 1 to criterion 2. For the input data $A, \boldsymbol{b}^1, \boldsymbol{b}^2$ and $\boldsymbol{b}^3$, CCM provides

$$\hat{\boldsymbol{b}}^1 = \begin{bmatrix} 0.21 \\ 0.79 \end{bmatrix}, \hat{\boldsymbol{b}}^2 = \begin{bmatrix} 0.52 \\ 0.48 \end{bmatrix}, \hat{\boldsymbol{b}}^3 = \begin{bmatrix} 0.83 \\ 0.17 \end{bmatrix}.$$

and

$$\boldsymbol{p} = \begin{bmatrix} 0.39 \\ 0.32 \\ 0.29 \end{bmatrix}.$$

From (2.4) and (2.5) we have

$$\boldsymbol{q} = \begin{bmatrix} 0.49 \\ 0.51 \end{bmatrix}. \quad (3.3)$$

The overall weight vector $\boldsymbol{q}$ of (3.3) means that criterion 2 is preferred to 1 in the aggregate. However, no alternative prefer criterion 2 to 1. This contradicts for Pareto principle that an overall ranking of criteria coincides with a ranking of criteria from any regulating alternative if all regulating alternatives have the same ranking of criteria (see the details for [4]).

In order to apply ANP to the numerical example (3.1) and (3.2), we have a supermatrix

$$S = \begin{bmatrix} 0 & 0 & 0.56 & 0.53 & 0.51 \\ 0 & 0 & 0.44 & 0.47 & 0.49 \\ 1/6 & 0.6 & 0 & 0 & 0 \\ 1/3 & 0.3 & 0 & 0 & 0 \\ 1/2 & 0.1 & 0 & 0 & 0 \end{bmatrix}$$

and find a principal eigenvector $\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$ of S. Since $\frac{\boldsymbol{x}}{\boldsymbol{e}^\top \boldsymbol{x}} = \begin{bmatrix} 0.53 \\ 0.47 \end{bmatrix}$ and $\frac{\boldsymbol{y}}{\boldsymbol{e}^\top \boldsymbol{y}} = \begin{bmatrix} 0.37 \\ 0.32 \\ 0.31 \end{bmatrix}$, the overall weight vector of criteria by ANP means that criterion 1 is preferred to 2 in the aggregate. Therefore, the overall weight vector of criteria by ANP satisfies Pareto principle. Though the ranking of alternatives by ANP is equal to that by CCM, the overall evaluation by ANP has more accountability than that by CCM from the viewpoint of Pareto principle. Furthermore, the following theorem guarantees that the overall weight vector of criteria by ANP always satisfies Pareto principle.

Theorem 3 *Let* $I = \{1, \ldots, m\}$ *and* $J = \{1, \ldots, n\}$*. Suppose that* $I = K$ *and that* $b^i_1 \le \cdots \le b^i_n$ *for all* $i \in I$*. Let* $[\boldsymbol{x}^\top, \boldsymbol{y}^\top]^\top$ *be a positive principal eigenvalue of a supermatrix*

$$\begin{bmatrix} 0 & \boldsymbol{b}^1 \cdots \boldsymbol{b}^m \\ A(\sum_{i\in I} A_i)^{-1} & 0 \end{bmatrix},$$

then $x_1 \le \cdots \le x_n$*.*

4. Avoiding the paradox of CCM

In order to recover a paradox of CCM as stated in section 3, we discuss the weighted averaging CCM as follows:

the weighted averaging CCM

Step 0: Choose a positive coefficient w_{ik} for all $k \in K$ and all $i \in I$ such that $\sum_{k\in K} w_{ik} = 1$ for all $i \in K$. For all $k \in K$, let

$$\boldsymbol{r}^k_0 := A_k^{-1}\boldsymbol{b}^k. \quad (4.1)$$

Set $t := 0$ and go to **Step 1**.

Step 1: For all $i \in I$, let

$$\boldsymbol{r}^i_{t+1} := \sum_{k\in K} w_{ik} \frac{\boldsymbol{r}^k_t}{\boldsymbol{e}^\top A_i \boldsymbol{r}^k_t}. \quad (4.2)$$

Step 2: If $\max_{k\in K} \left\| \boldsymbol{r}^k_{t+1} - \boldsymbol{r}^k_t \right\| = 0$, then let $\bar{\boldsymbol{r}}^k := \boldsymbol{r}^k_{t+1}$ for all $k \in K$ and stop.

Otherwise, set $t := t + 1$ and go to **Step 1**.

Let $\boldsymbol{r}^k = \boldsymbol{r}^k_0$ for all $k \in K$. Since there exists an regulating alternative l such that the ranking of $\{b^l_j \mid j \in J\}$ is the same as that of $\{r^l_j \mid j \in J\}$, we set $w_{il} \approx 1$ and $w_{ik} \approx 0$ for all $k \in K$ and all $i \in I$. Then, we may get $\{\hat{\boldsymbol{b}}^i \mid i \in K\}$ such that a ranking of $\{\hat{b}^i_j \mid j \in J\}$ is the same as that of $\{b^l_j \mid j \in J\}$ for all $i \in I$.

References

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