A Strange Case of the Cost and Allocative Efficiencies in DEA

01302170 National Graduate Institute for Policy Studies TONE Kaoru

1. Introduction

The concepts of cost (overall) and allocative efficiencies were first introduced by Farrell, and then developed by Färe, Grosskopf and Lovell by using linear programming technologies.

In this paper, we show that, in a single input case, the conventional cost efficiency is equal to the technical efficiency, and there is no room for adopting cost factors into the cost efficiency evaluation. Then, in a more general case, we demonstrate that, if two decision making units (DMUs) have the same amount of inputs and outputs, and one has unit-cost for inputs twice the other, then the two DMUs have the same cost (overall) and allocative efficiencies. After pointing out the irrationality of these efficiencies, we propose a new scheme that being free from the above shortcomings has several favorable properties.

2. Single Input Case

Theorem 1 *For the single input case, the technical efficiency θ^* is equal to the cost efficiency γ^*.*

Corollary 1 *In the single input case, the allocative efficiency is always one for every DMU.*

This sounds very strange, since, in this case, the input cost seems to have nothing to do with the allocative efficiency.

3. General Case

Here we observe a more general case where we have m inputs $(x_1, \ldots, x_m)$. Suppose that DMUs A and B have the same amount of inputs and outputs, i.e., $\boldsymbol{x}_A = \boldsymbol{x}_B$ and $\boldsymbol{y}_A = \boldsymbol{y}_B$. Assume further that the unit cost of DMU A is twice that of DMU B for each input, i.e., $\boldsymbol{c}_A = 2\boldsymbol{c}_B$. Under these assumptions, we have the following theorem:

Theorem 2 *Both DMUs A and B have the same cost (overall) and allocative efficiencies.*

This also sounds very strange, since DMUs A and B have the same cost and allocative efficiencies even though the cost of DMU B is half that of DMU A.

4. A New Scheme

The previous two sections reveal the shortcomings and irrationality of the cost and allocative efficiencies proposed thus far. These shortcomings are caused by the structure of the supposed production possibility set P as defined by:

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq X\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \boldsymbol{0}\}. \quad (1)$$

P is defined only by using technical factors $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in R^{m\times n}$ and $Y = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n) \in R^{s\times n}$, but has no concern with the unit input cost $C = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n)$.

Let us define an another cost-based production possibility set P_c as:

$$P_c = \left\{(\bar{\boldsymbol{x}}, \boldsymbol{y}) | \bar{\boldsymbol{x}} \geq \bar{X}\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \boldsymbol{0}\right\}, \quad (2)$$

where $\bar{X} = (\bar{\boldsymbol{x}}_1, \ldots, \bar{\boldsymbol{x}}_n)$ with $\bar{\boldsymbol{x}}_j = (c_{1j}x_{1j}, \ldots, c_{mj}x_{mj})^T$.

Here, we assume that the matrices X and C are non-negative, and all inputs are associated with cost, although we discuss the inclusion of non-cost inputs later. Also we assume that the elements of $\bar{x}_{ij} = (c_{ij}x_{ij})$ $(\forall(i,j))$ are denominated in homogeneous units, viz., dollars, so that adding up the elements of $\bar{x}_{ij}$ has a meaning. Though we only assume here the convexity for the sets P and P_c as defined respectively by (1) and (2), this convexity issue can be discussed.

Based on this new production possibility set P_c, a new technical efficiency $\bar{\theta}^*$ is obtained as the optimal solution of the following LP problem:

$$\begin{array}{llr} \text{[NTec]} & \bar{\theta}^* = \min\ \bar{\theta} & (3) \\ \text{subject to} & \bar{\theta}\bar{\boldsymbol{x}}_o \geq \bar{X}\boldsymbol{\lambda} & (4) \\ & \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} & (5) \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. & (6) \end{array}$$

The new cost efficiency $\bar{\gamma}^*$ is defined as

$$\bar{\gamma}^* = \boldsymbol{e}\bar{\boldsymbol{x}}_o^*/\boldsymbol{e}\bar{\boldsymbol{x}}_o, \quad (7)$$

where $\boldsymbol{e} \in R^m$ is a row vector with all elements being equal to 1, and $\bar{\boldsymbol{x}}_o^*$ is the optimal solution of the LP given below:

$$\begin{array}{llr} \text{[NCost]} & \min\ \boldsymbol{e}\bar{\boldsymbol{x}} & (8) \\ \text{subject to} & \bar{\boldsymbol{x}} \geq \bar{X}\boldsymbol{\lambda} & (9) \\ & \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} & (10) \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. & (11) \end{array}$$

Theorem 3 *The new cost efficiency $\bar{\gamma}^*$ is not greater than the new technical efficiency $\bar{\theta}^*$.*

5. Differences Between the Two Models

In the traditional model, keeping the unit cost of DMU$_o$ fixed at c_o, the optimal input mix $\boldsymbol{x}^*$ that produces the output $\boldsymbol{y}_o$ is found. As we observe in the third section, this model does not pay attention to possible choices of other unit costs.

In the new model, we search for the optimal input mix $\bar{\boldsymbol{x}}^*$ for producing $\boldsymbol{y}_o$ (or more). More concretely, the optimal mix is described as:

$$\bar{x}_i^* = \sum_{j=1}^{n} c_{ij} x_{ij} \lambda_j^*. \; (i = 1, \ldots, m) \qquad (12)$$

Hence, it is assumed that, for a given output $\boldsymbol{y}_o$, the optimal input mix can be found (and realized) independently of the current unit cost $\boldsymbol{c}_o$ of DMU$_o$.

These points are the fundamental differences between the two models. Using the traditional one we cannot recognize the existence of other cheaper input mix, as we have demonstrated in the third section.

Since the worldwide globalization of production has become a current trend, we should be able to find out the optimal input mix or, at least, to notify the existence of cheaper ones through the cost efficiency evaluation.

For comparisons of these two models, we demonstrate a simple example concerning three DMUs A, B and C with each using two inputs (x_1, x_2) to produce one output (y) along with input costs (c_1, c_2) as exhibited in Table 1.

Table 1: Comparison of Two Schemes

				Traditional				
	x_1	c_1	x_2	c_2	y	θ^*	γ^*	α^*
A	10	10	10	10	1	0.5	0.35	0.7
B	10	1	10	1	1	0.5	0.35	0.7
C	5	3	2	6	1	1	1	1
				New Scheme				
	$\bar{x}_1$	e_1	$\bar{x}_2$	e_2	y	$\bar{\theta}^*$	$\bar{\gamma}^*$	$\bar{\alpha}^*$
A	100	1	100	1	1	0.1	0.1	1
B	10	1	10	1	1	1	1	1
C	15	1	12	1	1	0.83	0.74	0.88

For DMUs A and B, the traditional model gives the same technical (θ^*), cost (γ^*) and allocative (α^*) efficiency scores, as expected from Theorem 2. DMU C is found to be the only best performer in this framework.

However, the new scheme distinguishes DMU A from DMU B by assigning them different technical and cost efficiency scores (See Table 1). This is due to the difference in their unit costs. Moreover, DMU B is judged as both technically and cost efficient due to its improvement in cost efficiency score from $0.35(\gamma_B^*)$ to $1(\bar{\gamma}_B^*)$. As is seen in Table 1, this cost difference brings out a drop in DMU A's cost efficiency score from $0.35(\gamma_A^*)$ to $0.1(\bar{\gamma}_A^*)$. Also, the optimal input mix is now changed: $x_1^* = 5$ and $x_2^* = 2$ in the old scheme, and $x_1^* = 10$ and $x_2^* = 10$ in the new scheme.

6. Validating the New Scheme

Firstly, we point out that, if the unit cost for inputs, $\boldsymbol{c} = (c_1, \ldots, c_m)$, is the same among all DMUs, the proposed new efficiencies are the same as the traditional ones as described in the second and third sections, and no strange phenomena occur in this case. However, it is quite usual that the unit costs of inputs such as labor, material and capital differ from one DMU to another. We now investigate several characteristics of the new measures.

7. On the Monotonicity of New Measures

Theorem 4 *If $\boldsymbol{x}_A = \boldsymbol{x}_B$, $\boldsymbol{y}_A = \boldsymbol{y}_B$ and $\boldsymbol{c}_A \geq \boldsymbol{c}_B$, then we have the following inequalities: $\bar{\theta}_A^* \leq \bar{\theta}_B^*$ and $\bar{\gamma}_A^* \leq \bar{\gamma}_B^*$. Furthermore, strict inequalities hold if $\boldsymbol{c}_A > \boldsymbol{c}_B$.*

Thus, the new measure helps to secure us against the occurring of the strange phenomenon observed in this paper.

8. Uses of the Two Technical Measures

We have two technical efficiency scores, θ^* and $\bar{\theta}^*$ for each DMU. The former is determined based only on purely technical input factors, while the latter is based on both input and cost factors. If, for a DMU, θ^* is low and $\bar{\theta}^*$ is high, this suggests the need for input reduction. On the other hand, if θ^* is high and $\bar{\theta}^*$ is low, the DMU needs an improvement in cost factors, i.e., cost reduction. Thus, both efficiency measures are utilized for characterizing the DMU and at the same time, suggest directions for improvement. Uses of these two measures prevent the occurring of strange case described in the second section.

References

1. Färe R, Grosskopf S and Lovell CAK (1985). *The Measurement of Efficiency of Production* Boston: Kluwer Nijhoff.

2. Färe R, Grosskopf S and Lovell CAK (1994). *Production Frontiers* Cambridge: Cambridge University Press.