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# Optimum Requirement Spanning Tree Problem via a Network Server

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## 1 Introduction

We introduce in this paper a variation of the optimum requirement spanning tree problem (ORSTP) first studied by Hu [3] in 1974.

ORSTP is often discussed as a problem of finding an 'optimum' communication network of tree type. In ORSTP, any two vertices are assumed to communicate directly with each other. Hu [3] considered the case when the underlying graph (say $G$) is not necessarily a complete graph, and showed that ORSTP can be solved by Gomory-Hu algorithm [2] which is originally an algorithm to solve the all-pairs network flow problem efficiently. Anazawa [1] considered a generalized ORSTP where $G$ is complete and maximum degree constraints are imposed, and gave a condition under which the problem is explicitly solvable.

On the other hand, we suppose in the new problem that a message from a vertex is transmitted to the other *via* the network server. Let $V_0 = \{0, 1, 2, \ldots, n\}$ (a vertex set), $V = V_0 \setminus \{0\}$, and suppose that a simple graph $G_0$ on $V_0$ is given. In the sequel of this paper, we regard vertex 0 and $V$ as a network server and a set of client nodes, respectively. For each pair of vertices $\{v, w\}$ $(v, w \in V)$, a nonnegative value $r_{vw}$ (called a *requirement*, representing the frequency of communication) is given. For a subgraph $G'$ of $G_0$, let $d(v, w; G')$ denote the number of edges forming the shortest path(s) between $v$ and $w$ on $G'$. Let $g(x, y)$ be an arbitrary bivariate function satisfying $g(x+1, y) \geq g(x, y)$ $(0 \leq x \leq n-1,\ 0 \leq y \leq n)$, $g(x, y+1) \geq g(x, y)$ $(0 \leq x \leq n,\ 0 \leq y \leq n-1)$, and $g(x, y) = g(y, x)$ $(0 \leq x, y \leq n)$. The new problem in this paper is of finding a spanning tree $T$ of $G_0$ to minimize

$$f_g(T) = \sum_{v \in V} \sum_{w \in V} g(d(v, 0; T), d(0, w; T)) r_{vw}.$$

And we consider the following two cases: (I) $G_0$ is not complete, and (II) $G_0$ is complete, but degrees of vertices are constrained.

## 2 Case I

Supposing that the given $G_0$ is not a complete graph, we consider the following:

**Problem 1** Find a spanning tree $T$ of $G_0$ which minimizes $f_g(T)$.

In this case, since an $f_g$-optimum tree can be obtained by Dijkstra's algorithm, we can obtain a simple result as follows.

**Theorem 1** *Problem 1 is solved in $O(n^2)$ time.*

## 3 Case II

Suppose that $G_0$ is complete and, for each vertex $v \in V_0$, the upper bound of degree (denoted by $l_v$) is given. Also, suppose that $\{l_v\}$ satisfies $1 \leq l_0 < n$, $n > l_1 \geq l_2 \geq \cdots \geq l_n \geq 1$, and $\sum_{v=0}^{n} l_v \geq 2n$. Then the problem to be discussed here is expressed as follows.

**Problem 2** Find a spanning tree $T$ of $G_0$ to minimize $f_g(T)$ subject to

$$\deg(v; T) \leq l_v \ (v \in V_0). \tag{1}$$

In the rest of this paper, we regard all spanning trees of $G_0$ as rooted trees with root 0 $(\in V_0)$. For a spanning tree $T$ of $G_0$, let $\pi(x; T)$ be the parent of $x$, and $\chi(x; T)$ the set of children of $x$.

For a spanning tree $T = (V, E)$ of $G_0$ satisfying condition (1), we define two kinds of transformation of $T$ which may reduce the $f_g$ value as follows.

**Raising:** For a vertex $x \in V$ with $d(0, x; T) = h \geq 2$, if there exists a vertex $p \in V_0$ with $d(0, p; T) \leq h - 2$ and $\deg(p; T) < l_p$, then make a tree $T' = (V, E')$ such that

$$E' = E \setminus \{(\pi(x; T), x)\} \cup \{(p, x)\}.$$

**Exchanging:** For two vertices $x, y \in V$ satisfying $x < y$ and $d(0, x; T) \geq d(0, y; T)$,

(i) if $\deg(x; T) \leq l_y$, then make a tree $T' = (V, E')$ such that

$$\begin{aligned} E' = \\ & E \setminus (\{(\pi(x; T), x)\} \cup \{(x, v) | v \in \chi(x; T)\}) \\ & \setminus (\{(\pi(y; T), y)\} \cup \{(y, v) | v \in \chi(y; T)\}) \\ & \cup (\{(\pi(x; T), y)\} \cup \{(y, v) | v \in \chi(x; T)\}) \\ & \cup (\{(\pi(y; T), x)\} \cup \{(x, v) | v \in \chi(y; T)\})\,; \end{aligned} \tag{2}$$

(ii) if $l_y < \deg(x; T) \leq l_x$, then let $\chi'(x; T)$ be an arbitrary subset of $\chi(x; T)$ with $|\chi'(x; T)| = \deg(x; T) - l_y$, let $\bar{\chi}(x; T) = \chi(x; T) \setminus \chi'(x; T)$, and make a tree $T' = (V, E')$ such that $\chi(x; T)$ in the right hand side of equation (2) is replaced by $\bar{\chi}(x; T)$.

**Lemma 1** *For a spanning tree $T$ of $G_0$ satisfying condition (1), let $T'$ be obtained by applying one raising to $T$. Then $f_g(T) \geq f_g(T')$ holds.*

**Lemma 2** *For a spanning tree $T$ of $G_0$ satisfying condition (1), let $T'$ be obtained by applying one exchanging to $T$. If $\{r_{vw}\}$ satisfies*

$$r_{xx} \geq r_{yy} \ (1 \leq x < y \leq n), \text{ and} \tag{3}$$

$$r_{xw}+r_{wx} \geq r_{yw}+r_{wy} \ (1 \leq x < y \leq n, \ w \in V \setminus \{x,y\}), \quad (4)$$

*then* $f_g(T) \geq f_g(T')$ *holds.*

**Corollary 1** *Suppose that the bivariate function* $g$ *is expressed as*

$$g(x,y) = g_1(x) + g_1(y), \quad (5)$$

*where* $g_1(x)$ *is an arbitrary monotone nondecreasing function for* $x \in [0,n]$. *For a spanning tree* $T$ *of* $G_0$ *satisfying condition (1), let* $T'$ *be obtained by applying one exchanging to* $T$. *If* $\{r_{vw}\}$ *satisfies*

$$\sum_{v\in V} r_{xv} + \sum_{v\in V} r_{vx} \geq \sum_{v\in V} r_{yv} + \sum_{v\in V} r_{vy} \ (1 \leq x < y \leq n), \quad (6)$$

*then* $f_g(T) \geq f_g(T')$ *holds.*

Let $D_0 = \{0\}$, $D_1 = \{1,2,\ldots,l_0\}$, and $D_h = \{\max D_{h-1} + i | i = 1,2,\ldots,\sum_{v\in D_{h-1}}(l_v - 1)\}$ $(h = 2,3,\ldots,H)$, where $H$ is the smallest integer satisfying $\max D_H \geq n$. We consider a class of spanning trees satisfying (1) and

$$d(0,v;T) = h \ (v \in D_h \cap V_0). \quad (7)$$

Note that any spanning tree of $G_0$ can be transformed to a tree with (1) and (7) by applying raising and exchanging procedures finite times.

**Theorem 2** *Let* $T^*$ *be a spanning tree of* $G_0$ *satisfying properties (1) and (7). If* (a) $\{r_{vw}\}$ *satisfies conditions (3) and (4), or* (b) $g$ *is expressed as equation (5) and* $\{r_{vw}\}$ *satisfies condition (6), then* $f_g(T^*) \leq f_g(T)$ *holds for any spanning tree* $T$ *of* $G_0$.

Since there exists an $O(n)$-time algorithm for constructing a spanning tree of $G_0$ with properties (1) and (7), we obtain the following:

**Theorem 3** *If* (a) $\{r_{vw}\}$ *satisfies conditions (3) and (4), or* (b) $g$ *is expressed as equation (5) and* $\{r_{vw}\}$ *satisfies condition (6), then Problem 2 is solved in* $O(n)$ *time.*

# 4 Applications

One of the typical situations to be modeled by the problem in this paper is the construction of a local area network (LAN) of tree type in which an e-mail delivery system works 'optimally' in a certain sense. Here, we consider two criterions of optimality.

**Reliability of the delivery of messages:** Under the condition that a message is sent by a certain vertex in a given spanning tree $T$ of $G_0$, let $p(T)$ be the conditional probability that the message is received by its destination vertex without any delay due to network failures. Under some assumptions, we find that

$$1 - p(T) = \sum_{v\in V}\sum_{w\in V} h(d(v,0;T), d(0,w;T))r_{vw},$$

where $h(x,y) = 1 - h_1(x)h_1(y)$ and

$$h_1(x) = \sum_{k=0}^{2n+1}\sum_{i+j=k}\binom{n-x}{i}\binom{n-x}{j} \times p^i(1-p)^{n-i}q^j(1-q)^{n+1-j}$$

($p$ and $q$ denote the probabilities of edge and vertex failures, respectively). Hence, we find from Theorem 2 that, if $\{r_{vw}\}$ satisfies conditions (3) and (4), then any spanning tree $T$ with properties (1) and (7) minimizes $1 - p(T)$ (that is, maximizes $p(T)$).

**Security against unauthorized access:** Let $L$ be a random variable denoting the length of the unique walk via the e-mail server passed by a message from a certain vertex to another one. To keep the damage by unauthorized access at the minimum, we should design networks to minimize the expectation of $L$, denoted by $E(L)$. Let $r_{vw}$ denote the frequency of sending a message from $v$ to $w$. Then we have

$$E(L) = \sum_{v\in V}\sum_{w\in V} g(d(v,0;T), d(0,w;T))r_{vw},$$

where $g(x,y) = x + y$. Therefore, we find from Theorem 2 that, if $\{r_{vw}\}$ satisfies condition (6), then any spanning tree $T$ with properties (1) and (7) minimizes $E(L)$.

# References


[1] T. Anazawa: A Generalized Optimum Requirement Spanning Tree Problem with a Monge-like Property, *Journal of the Operations Research Society of Japan,* **43** (2000) 417-428.

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