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Pairwise Stability in a General Two-Sided Matching Model Based on Discrete Concave Utility Functions

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1 Introduction

The marriage model due to Gale and Shapley [2] and the assignment model due to Shapley and Shubik [4] are standard in the theory of two-sided matching markets. We give a common generalization of these models by utilizing M$^\natural$-concave utility functions and show the existence of a pairwise stable outcome in our general model. Our present model is a further natural extension of the general model examined in our previous paper [1].

2 M$^\natural$-Concavity

We briefly explain the concept of M$^\natural$-concave functions (see [3] for details). Let E be a nonempty finite set, and let $\mathbf{Z}$ and $\mathbf{R}$ be the sets of integers and reals, respectively. We denote by $\mathbf{Z}^E$ the set of integer vectors $x = (x(e) \mid e \in E)$ indexed by E, where $x(e)$ denotes the eth component of vector x. Also, $\mathbf{R}^E$ denotes the set of real vectors indexed by E. Let $\mathbf{0}$ be a vector of all zeros of an appropriate dimension. For each $e \in E$, we denote by χ_e the characteristic vector of e defined by: $\chi_e(e) = 1$ and $\chi_e(e') = 0$ for $e' \neq e$. For a vector $p \in \mathbf{R}^E$ and a function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$, we define the function $f[p]$ in $x \in \mathbf{Z}^E$ by

$$f[p](x) = f(x) + \sum_{e \in E} p(e)x(e).$$

We also define the set of maximizers of f on $U \subseteq \mathbf{Z}^E$ and the *effective domain* of f by

$$\begin{aligned} &\arg\max\{f(y) \mid y \in U\} = \\ &\qquad \{x \in U \mid \forall y \in U: \ f(x) \geq f(y)\}, \\ &\mathrm{dom}\, f = \{x \in \mathbf{Z}^E \mid f(x) > -\infty\}. \end{aligned}$$

A function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M$^\natural$-concave* if it satisfies[1]

(M$^\natural$) $\forall x, y \in \mathrm{dom}\, f$, $\forall e \in \{i \mid x(i) > y(i)\}$, $\exists e' \in \{i \mid x(i) < y(i)\} \cup \{0\}$:

$$f(x) + f(y) \leq f(x - \chi_e + \chi_{e'}) + f(y + \chi_e - \chi_{e'}),$$

where 0 is a new element not in E and χ_0 is a zero vector in $\mathbf{Z}^E$.

If f is M$^\natural$-concave, then $f[p]$ is also M$^\natural$-concave for any $p \in \mathbf{R}^E$.

3 Model Description

We consider a two-sided market consisting of disjoint sets M and W of agents, in which an agent in M may be called a worker and one in W a firm. Each worker $i \in M$ can supply multi-units of labor time, and each firm $j \in W$ can employ workers with multi-units of labor time and pay a salary to worker i per unit of labor time if j hires i. We assume that each pair (i, j) has lower and upper bounds on a salary per unit of labor time. We will discuss pairwise stability in this market.

Let $E = M \times W$, i.e., the set of all pairs (i, j) of agents $i \in M$ and $j \in W$. Also define $E_{(i)} = \{i\} \times W$ for all $i \in M$ and $E_{(j)} = M \times \{j\}$ for all $j \in W$. Denoting by $x(i, j)$ the number of units of labor time for which j hires i, we represent a labor allocation by vector $x = (x(i, j) \mid (i, j) \in E) \in \mathbf{Z}^E$. We express lower and upper bounds of salaries by two vectors $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ with $\underline{\pi} \leq \overline{\pi}$. For $y \in \mathbf{R}^E$ and $k \in M \cup W$, we denote by $y_{(k)}$ the restriction of y on $E_{(k)}$. We assume that a utility (in monetary terms) of each agent $k \in M \cup W$ is described by a function $f_k : \mathbf{Z}^{E_{(k)}} \to \mathbf{R} \cup \{-\infty\}$, and furthermore, that f_k satisfies the following assumption:

(A) $\mathrm{dom}\, f_k$ is bounded and hereditary, and has $\mathbf{0}$ as the minimum point,

[1] Condition (M$^\natural$) is denoted by ($-$M$^\natural$-EXC) in [3].

where heredity means that for any $y, y' \in \mathbf{Z}^{E(k)}$, $\mathbf{0} \leq y' \leq y \in \operatorname{dom} f_k$ implies $y' \in \operatorname{dom} f_k$.

A vector $x \in \mathbf{Z}^E$ is called a *feasible allocation* if $x_{(k)} \in \operatorname{dom} f_k$ for all $k \in M \cup W$. Given a feasible allocation x, a vector $s \in \mathbf{R}^E$ is called an *x-compatible salary vector* if $\underline{\pi}(i,j) \leq s(i,j) \leq \overline{\pi}(i,j)$ for all $(i,j) \in E$ with $x(i,j) > 0$ and if $s(i,j) = 0$ for all $(i,j) \in E$ with $x(i,j) = 0$. We call a pair (x,s) of a feasible allocation $x \in \mathbf{Z}^E$ and an x-compatible salary vector $s \in \mathbf{R}^E$ an *outcome.* An outcome (x,s) is said to be *individually rational* if

$$
\begin{aligned}
f_i[+s_{(i)}](x_{(i)}) &= \max\{f_i[+s_{(i)}](y) \mid y \leq x_{(i)}\} \\
&\qquad\qquad (\forall i \in M), \quad (1) \\
f_j[-s_{(j)}](x_{(j)}) &= \max\{f_j[-s_{(j)}](y) \mid y \leq x_{(j)}\} \\
&\qquad\qquad (\forall j \in W). \quad (2)
\end{aligned}
$$

Conditions (1) and (2) mean that each agent has no incentive to decrease labor time for the current salaries. For $s \in \mathbf{R}^E$, $\alpha \in \mathbf{R}$, $i \in M$ and $j \in W$, let $(s_{(i)}^{-j}, \alpha)$ be defined as the vector obtained from $s_{(i)}$ by replacing its jth component by α, and $(s_{(j)}^{-i}, \alpha)$ be similarly defined. An outcome (x,s) is called *pairwise unstable* if it is not individually rational or there exist $i \in M$, $j \in W$, $\alpha \in [\underline{\pi}(i,j), \overline{\pi}(i,j)]$, $y' \in \mathbf{Z}^{E(i)}$ and $y'' \in \mathbf{Z}^{E(j)}$ such that

$$
\begin{aligned}
f_i[+s_{(i)}](x_{(i)}) &< f_i[+(s_{(i)}^{-j}, \alpha)](y'), && (3) \\
y'(i,j') &\leq x(i,j') \quad (\forall j' \in W \backslash j), && (4) \\
f_j[-s_{(j)}](x_{(j)}) &< f_j[-(s_{(j)}^{-i}, \alpha)](y''), && (5) \\
y''(i',j) &\leq x(i',j) \quad (\forall i' \in M \backslash i), && (6) \\
y'(i,j) &= y''(i,j). && (7)
\end{aligned}
$$

Conditions (3)$\sim$(7) say that i and j can strictly increase their utilities by concertedly changing the current salary and labor time between them under the constraints that units of labor time of the other parts are not increased. An outcome (x,s) is called *pairwise stable* if it is not pairwise unstable. We also consider a stronger pairwise stability. We say that an outcome (x,s) is *pairwise quasi-unstable* if it is not individually rational or there exist $i \in M$, $j \in W$, $\alpha \in [\underline{\pi}(i,j), \overline{\pi}(i,j)]$, $y' \in \mathbf{Z}^{E(i)}$ and $y'' \in \mathbf{Z}^{E(j)}$ satisfying (3)$\sim$(6) (but not necessarily (7)). Trivially, a pairwise unstable outcome is pairwise quasi-unstable. An outcome (x,s) is called *pairwise strictly stable* if it is not pairwise quasi-unstable. Thus, an outcome (x,s) is pairwise strictly stable if and only if (1) and (2) hold and for all $i \in M$, $j \in W$ and $\alpha \in \mathbf{R}$ with $\underline{\pi}(i,j) \leq \alpha \leq \overline{\pi}(i,j)$,

$$
\begin{aligned}
f_i[+s_{(i)}](x_{(i)}) \geq \max\{f_i[+(s_{(i)}^{-j}, \alpha)](y) \mid \\
y(i,j') \leq x(i,j'),\ \forall j' \neq j\},
\end{aligned} \quad (8)
$$

or

$$
\begin{aligned}
f_j[-s_{(j)}](x_{(j)}) \geq \max\{f_j[-(s_{(j)}^{-i}, \alpha)](y) \mid \\
y(i',j) \leq x(i',j),\ \forall i' \neq i\}.
\end{aligned} \quad (9)
$$

Note that any pairwise strictly stable outcome is pairwise stable.

The concept of pairwise stability in our model coincides, in special cases, with those in the marriage model due to Gale and Shapley [2], the assignment model due to Shapley and Shubik [4], and so on.

The following theorem states that the existence of a pairwise strictly stable outcome, and hence, the existence of a pairwise stable outcome, is certified by M$^\natural$-concave utility functions.

Theorem 3.1 *For M$^\natural$-concave functions f_k ($k \in M \cup W$) satisfying* (A) *and for all vectors $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ with $\underline{\pi} \leq \overline{\pi}$, there exists a pairwise strictly stable outcome (x,s), and hence, there exists a pairwise stable outcome.*

References

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