日本オペレーションズ・リサーチ学会
2004年秋季研究発表会

Applying Path Counting Methods for Evaluating Edge and Node Deletion Connectivity Functions for the Network-structured System

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1. Introduction

There are many important and indispensable network-structured systems surrounding our daily lives, such as road networks, electric power networks, city gas networks, water pipeline network, and so on. These networks are generally called 'lifeline' networks needed to maintain our daily lives. This paper aims at evaluating quantitatively the reliability, the stability, and the strength of the connectedness of these network-structured systems.

2. Connectivity function of the network-structured systems

Let the undirected network $N = (V, E)$ consist of vertex set V and edge set E with cardinality $|V| = n$ and $|E| = m$, respectively. In the network $N = (V, E)$ there exist $\frac{n(n-1)}{2}$ paths connecting two different nodes in case we do not care about the direction. Suppose that k edges out of m edges in the network $N = (V, E)$ be deleted, then we count the number of paths connecting two different vertices in the obtained network. Let the number of paths connecting two different vertices be denoted by $c_m(N, k)$, and let the ratio between $c_m(N, k)$ and $c_m(N, 0)$ be $s_m(N, k)$, i.e., we define the following function for all $k \in K = \{1, 2, \cdots, m\}$.

$$s_m(N,k) = \frac{c_m(N,k)}{c_m(N,0)} \qquad k \in K \tag{1}$$

The total number of paths connecting two different nodes when none of the edges is deleted is given by $c_m(N, 0) = \frac{n(n-1)}{2}$, thus regarding as $c_m(N, k)$, we obtain the following relation.

$$0 \le s_m(N,k) = \frac{2c_m(N,k)}{n(n-1)} \le 1 \qquad k \in K \tag{2}$$

Similarly, deleting p nodes out of n nodes in the network, we count the number of paths connecting two different vertices in the obtained network. Let the number of paths connecting two different vertices be denoted by $d_n(N, p)$, and let the ratio between $d_n(N, p)$ and $d_n(N, 0)$ be $t_n(N, p)$, i.e., we define the following function for all $p \in L = \{1, 2, \cdots, n\}$.

$$t_n(N,p) = \frac{d_n(N,p)}{d_n(N,0)} \qquad p \in L \tag{3}$$

The total number of paths connecting two different nodes when none of the nodes is deleted is given by $d_n(N, 0) = \frac{n(n-1)}{2}$, thus regarding as $t_n(N, p)$, we obtain the following relation.

$$0 \le t_n(N,p) = \frac{2d_n(N,p)}{n(n-1)} \le 1 \qquad p \in L \tag{4}$$

Note that in general neither $c_m(N, k)$, $s_m(N, k)$ nor $d_n(N, p)$, $t_n(N, p)$ can be unique, i.e., given the network $N = (V, E)$ neither function $S_m(N, k) = \{s_m(N, k)\}$ nor $T_n(N, p) = \{t_n(N, p)\}$ can be unique with respect to k and p.

When we delete k edges out of m edges or p nodes out of n nodes in the network $N = (V, E)$, resulting values $s_m(N, k)$ or $t_n(N, p)$ differ according to how we delete these k edges or p nodes. We call functions $s_m(N, k)$ and $t_n(N, p)$ **edge deletion connectivity function** and **node deletion connectivity function**, respectively.

Let the maximum and the minimum for each k in the function $s_m(N, k)$ be denoted by $\overline{s_m}(N, k)$ and $\underline{s_m}(N, k)$, respectively.

$$\begin{aligned} \overline{s_m}(N,k) &= \max\{s_m(N,k)\} \\ \underline{s_m}(N,k) &= \min\{s_m(N,k)\} \end{aligned}$$

Similarly, let the maximum and the minimum for each p in the function $t_m(N, p)$ be denoted as follows.

$$\begin{aligned} \overline{t_n}(N,p) &= \max\{t_n(N,p)\} \\ \underline{t_n}(N,p) &= \min\{t_n(N,p)\} \end{aligned}$$

3. Properties for the edge deletion connectivity function

Let the graphs, such as a single path graph, a star graph and a circuit graph be denoted by P_m, W_m and C_m, respectively. Then we obtain the following theorem.

Theorem 1 The followings hold for the graphs P_m, W_m, and C_m, respectively.

$$\overline{s_m}(P_m,k) = \frac{(k-m)(k-m-1)}{m(m+1)} \tag{5}$$

$$\underline{s_m}(P_m,k) = \begin{cases} \frac{2(m-k)}{m(m+1)} & k \ge \frac{m-1}{2} \\ \frac{(\alpha_{mk}+1)(\beta_{mk}+m-k)}{m(m+1)} & k < \frac{m-1}{2} \end{cases} \tag{6}$$

$$s_m(W_m,k) = \overline{s_m}(P_m,k) \tag{7}$$

$$\overline{s_m}(C_m,k) = \begin{cases} 1 & k=1 \\ \overline{s_{m-1}}(P_{m-1},k-1) & k \ge 2 \end{cases} \tag{8}$$

$$\underline{s_m}(C_m,k) = \begin{cases} 1 & k=1 \\ \underline{s_{m-1}}(P_{m-1},k-1) & k \ge 2 \end{cases} \tag{9}$$

where α_{mk} and β_{mk} are given as follows.

$$\alpha_{mk} = \lfloor \frac{m-k}{k+1} \rfloor, \quad \beta_{mk} = m - k - (k+1)\alpha_{mk}.$$

The relations (8) and (9) are based upon the fact that a single path consisting of $m-1$ edges is obtained when we delete one edge from m edges, and the graph is connected. When we delete two or more than two edges out of m edges, a similar case as a single path occurs.

4. Properties for the node deletion connectivity function

We consider the maximum and the minimum for the connectivity function after deleting several nodes in the network $N = (V, E)$. It is obvious that we have $t_n(N, n-1) = 0$ regarding as the minimum. Next, deleting p nodes, number of paths connecting any two nodes in the network decreases by the number of paths connecting the deleted nodes with all other nodes, $p(n-p)+p(p-1)/2 = p(2n-p-1)/2$. Thus the following holds.

$$\overline{t_n}(N,p) \leq 1 - \frac{p(2n-p-1)}{n(n-1)} \tag{10}$$

Therefore, we obtain the following theorem regarding as whether the equality holds or not in the above relation.

Theorem Deleting p nodes from the network $N = (V, E)$, the necessary and sufficient condition for the relation

$$t_n(N,p) = 1 - \frac{p(2n-p-1)}{n(n-1)} \tag{11}$$

to hold is that the resulting network with $n-p$ nodes obtained after deleting p nodes is connected.

To prove the above theorem we need to show that the difference between the total number of paths between any two nodes and the number of paths decreased by deleting nodes is given by the number of paths in the resulting connected network consisting of $n-p$ nodes. Actually we can show the following relation holds.

$$\frac{n(n-1)}{2} - \frac{p(2n-p-1)}{2} = \frac{(n-p)(n-p-1)}{2} \tag{12}$$

Therefore, we can show that deleting p nodes in the connected network $N = (V, E)$, the resulting network remains connected when we follow the following node deletion rule.

Node deletion rule

1. Delete the node when there exists a node with degree 1.
2. When there is no node with degree 1, delete the node with smallest degree, which is not an isthmus.

Iterating the above two steps, we always obtain a connected network. Thus we obtain the following theorem.

Theorem 3 In the connected network $N = (V, E)$ the followings hold.

$$\overline{t_n}(N,p) = 1 - \frac{p(2n-p-1)}{n(n-1)} \tag{13}$$

Deleting p nodes from the networks $N = (V, E)$ such as P_m, W_m, and C_m, minimum of the connectivity function is obtained as follows.

Theorem 4 The followings hold for the graphs P_m, W_m, and C_m.

$$\overline{t_n}(W_m,p) = \overline{t_n}(P_m,p) \tag{14}$$

$$\overline{t_n}(C_m,p) = \begin{cases} 1 & p=1 \\ \overline{t_n}(P_{m-1},p-1) & p \geq 2 \end{cases} \tag{15}$$

5. Application

The quantitative evaluation method proposed in this paper for measuring the connectedness of a given network can be applicable for measuring the strength of the connectedness for the actual 'lifeline' network structured systems like electricity, gas, water supply, road and information communication. Namely, when some components of the network structured system become unavailable after breaking down, we try to measure quantitatively the stable connectedness of the network and evaluate how it changes.

We will present some numerical results on the proposed methods applied to actual networks and compare the property of edge deletion connectivity function and node deletion connectivity function.

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