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Interior Point Methods for the Monotone Linear Complementarity Problem in Symmetric Matrices

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1. Introduction.

We use the following notation and symbols:
$\hat{\mathcal{S}}$: the set of all $n \times n$ real matrices,
$\mathcal{S}$: the set of all $n \times n$ symmetric matrices,
$\mathcal{S}_+ = \{X \in \mathcal{S} : X \succeq O\}$,
$\mathcal{S}_{++} = \{X \in \mathcal{S} : X \succ O\}$,
$\hat{\mathcal{S}}_{++} = \{X \in \hat{\mathcal{S}} : X \succ O\}$,
Tr X : the trace of $X \in \hat{\mathcal{S}}$,
$\|X\|_F = \left(\mathrm{Tr}\ X^T X\right)^{1/2}$,
$\mathcal{F}$: an $n(n+1)/2$ dim. affine subspace of $\mathcal{S}^2$,
$\mathcal{F}_+ = \{(X, Y) \in \mathcal{F} : X \succeq O,\ Y \succeq O\}$,
$\mathcal{F}_{++} = \{(X, Y) \in \mathcal{F} : X \succ O,\ Y \succ O\}$,
$\mathcal{F}^* = \{(X, Y) \in \mathcal{F}_+ : \mathrm{Tr}\ XY = 0\}$.
$\hat{\mathcal{F}} = \left\{(X, Y) \in \mathcal{S} \times \hat{\mathcal{S}} : \left(X, \frac{Y+Y^T}{2}\right) \in \mathcal{F}\right\}$.

The purpose of this paper is to establish a general theoretical framework of interior-point methods for the monotone linear complementarity problem (LCP) in symmetric matrices. The LCP in symmetric matrices is the problem of finding an $(X, Y) \in \mathcal{F}$ such that

$$X \succeq O,\ Y \succeq O \text{ and } \mathrm{Tr}\ XY = 0. \tag{1}$$

We impose an assumption on the LCP (1).

Condition 1.1. $\mathcal{F}$ is monotone, *i.e.*, Tr $(X' - X)^T(Y' - Y) \geq 0$ for every (X', Y') and $(X, Y) \in \mathcal{F}$.

2. Some Basic Results.

2.1. The Central Trajectory.

Suppose that the LCP (1) has an interior feasible solution. Then, for every $\mu > 0$, there exists a unique $(X(\mu), Y(\mu)) \in \mathcal{F}_{++}$ such that $X(\mu)Y(\mu) = \mu I$. (For the proof, see [2].) We call $\mathcal{C} = \{(X(\mu), Y(\mu)) : \mu > 0\}$ the central trajectory.

2.2. Newton Directions toward the Central Trajectory.

Let $(X, Y) \in \mathcal{S}^2_{++}$ and $\mu = \mathrm{Tr}\ XY/n$. Choose $\beta > 0$. It might seem natural to regard the system of linear equations

$$(X+U, Y+V) \in \mathcal{F} \text{ and } UY+XV = \beta\mu I - XY \tag{2}$$

in variable matrices $U,\ V \in \mathcal{S}$ as the Newton equation at $(X, Y) \in \mathcal{S}^2_{++}$ for approximating a point $(X', Y') = (X+U, Y+V) \in \mathcal{S}^2_{++}$ on the central trajectory that satisfies

$$(X', Y') \in \mathcal{F} \text{ and } X'Y' = \beta\mu I. \tag{3}$$

However the system (2) does not necessarily have a solution ([2]). Hence we need a suitable modification in the systems (2) and (3) to consistently define Newton directions toward the central trajectory. So we consider the Newton equation at $(X, Y) \in \mathcal{S}_{++} \times \hat{\mathcal{S}}_{++}$ for approximating a point $(X', Y') = (X+U, Y+V)$ on the central trajectory which satisfies :

$$(X+U, Y+\hat{V}) \in \hat{\mathcal{F}} \text{ and } X\hat{V}+UY = \beta\mu I - XY \tag{4}$$

in variable matrices $U \in \hat{\mathcal{S}}$ and $\hat{V} \in \hat{\mathcal{S}}$. Then we have:

Theorem 2.1.

(i) *$\hat{\mathcal{F}}$ is monotone.*

(ii) *(X', Y') is a solution of the system (3) of equations if and only if it is a solution of $(X', Y') \in \hat{\mathcal{F}}$ and $X'Y' = \beta\mu I$.*

(iii) *Let $(X, Y) \in \mathcal{S}_{++} \times \hat{\mathcal{S}}_{++}$, $\mu = \mathrm{Tr}\ XY/n$ and $\beta \geq 0$. Then the Newton equation (4) has a unique solution $(U, \hat{V})$.*

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2.3. A Generic IP Method.

Now we are ready to describe a generic interior-point method.

Generic IP Method.

Step 0: Choose $(X^0, Y^0) \in \mathcal{S}^2_{++}$. Let $r = 0$.

Step 1: Let $(X, Y) = (X^r, Y^r)$ and $\mu = \frac{\mathrm{Tr}\, XY}{n}$.

Step 2: Choose *a direction parameter* $\beta \geq 0$.

Step 3: Compute a solution $(U, \hat{V}) \in \mathcal{S} \times \hat{\mathcal{S}}$ of the system (4) of equations.

Step 4: Let $V = (\hat{V} + \hat{V}^T)/2$.

Step 5: Choose *a step size parameter* $\alpha \geq 0$ such that

$$(\bar{X}, \bar{Y}) = (X, Y) + \alpha(U, V) \in \mathcal{S}^2_{++}. \quad (5)$$

Let $(X^{r+1}, Y^{r+1}) = (\bar{X}, \bar{Y})$.

Step 6: Replace $r + 1$ by r, and go to Step 1.

3. Some Interior-Point Methods.

In this section we present two types of interior-point methods, a central trajectory following method, a potential reduction method as special cases of the Generic IP Method. (See [2] for an infeasible-interior-point potential-reduction method.)

3.1. A Central Trajectory Following Method.

First we introduce a horn neighborhood of the central trajectory
$\mathcal{N}(\gamma) = \{(X, Y) \in \mathcal{F}_{++} : \|\sqrt{X}^T Y \sqrt{X} - \mu I\|_F \leq \gamma\mu,$ *where* $\mu = \frac{\mathrm{Tr}\, XY}{n}\}$.

Theorem 3.1. *Let* $\gamma \in (0,\ 0.1]$. *Suppose that* $(X, Y) \in \mathcal{N}(\gamma)$. *Let* $\beta = 1 - \gamma/\sqrt{n}$ *in Step 2 and* $\alpha = 1$ *in Step 5. Let* $\bar{\mu} = Tr\, \bar{X}\bar{Y}/n$. *Then*

$$(\bar{X}, \bar{Y}) = (X, Y) + (U, V) \in \mathcal{N}(\gamma),$$
$$\beta\mu \leq \bar{\mu} \leq \left(1 - \frac{\gamma}{2\sqrt{n}}\right)\mu.$$

Let $\epsilon > 0$. In view of the theorem above, if $r \geq \frac{2\sqrt{n}}{\gamma} \log \frac{\mathrm{Tr}\, X^0 Y^0}{\epsilon}$, then (X^r, Y^r) gives an approximate solution of the LCP (1) such that

$$(X^r, Y^r) \in \mathcal{F}_{++}, \mathrm{Tr}\, X^r Y^r \leq \epsilon. \quad (6)$$

3.2. A Potential-Reduction Method.

For every $(X, Y) \in \mathcal{F}_{++}$, define the potential function $f(X, Y) = (n + \nu) \log \mathrm{Tr}\, XY - \log\det XY - n \log n$. Here $\nu \geq 0$ is a parameter. Let

$$\left.\begin{array}{rcl} H(\beta) & = & \beta\mu\sqrt{X}^{-1}\sqrt{Y}^{-T} - \sqrt{X}^T\sqrt{Y}, \\ \lambda_{min} & = & \min\{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \end{array}\right\} \quad (7)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of the matrix XY.

Theorem 3.2. *Let* $n \geq 3$, $\nu = \sqrt{n}$, $\tau = 0.4$ *and* $\delta = 0.2$. *Suppose that* $(X, Y) \in \mathcal{F}_{++}$. *Let* $\beta = n/(n + \nu)$ *in Step 2 and* $\alpha = \tau\sqrt{\lambda_{min}}/\|H(\beta)\|_F$ *in Step 5, where* τ, λ_{min} *and* $H(\beta)$ *are given in (7). Then* $(\bar{X}, \bar{Y}) = (X, Y) + \alpha(U, V) \in \mathcal{F}_{++}$, $f(\bar{X}, \bar{Y}) \leq f(X, Y) - \delta$.

By Theorem 3.2, if $r \geq \frac{f(X^0, Y^0) - \sqrt{n}\log\epsilon}{\delta}$ then (X^r, Y^r) gives an approximate solution of the LCP (1) satisfying (6).

References

[1] S. Boyd, L. E. Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, 1994.

[2] M. Kojima, S. Shindoh and S. Hara, "Interior-point methods for the monotone linear complementarity problem in symmetric matrices," 1994.

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