1995年度日本オペレーションズ・リサーチ学会
春季研究発表会

On Restricting Virtual Multipliers in Cone-Ratio DEA

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1. Introduction

Data Envelopment Analysis (DEA) has been widely applied for evaluating the relative efficiency of decision making units (DMUs) with multiple inputs and outputs. The relative efficiency is measured by a ratio scale of the virtual input *vs.* the virtual output, which are the weighted sums of inputs and outputs, respectively. Since the original Charnes, Cooper and Rhodes (CCR) model, many studies have been developed to cope with the actual situations of the problems. One of them is directed to research in the feasible region of the weights and has actually imposed some additional constraints to the weights. Representatively, such studies resulted in the Assurance Region (AR) model and the Cone-Ratio (CR) model. The assurance region method confines the feasible region of the weights by imposing a lower and an upper bounds to the ratio of some selected pairs of weights. On the other hand, the cone-ratio model solves the CCR model first and chooses a few exemplary efficient DMUs from among all the efficient ones by consulting with experts on the problem. Then, the corresponding optimal weights to the selected efficient DMUs are used to construct a convex-cone as the feasible region of the weights. However, usually the optimal weights are not uniquely determined and hence there is ambiguity in selecting the weights to form the convex cone.

In an effort to overcome this problem, this paper will propose three practical methods for deciding the convex cone in accordance with three principles which will be explained later.

2. Cell Subdivision of Multiplier Simplex

Suppose there are n DMUs with m inputs and s outputs. The i-th input and the r-th output of the j-th DMU are denoted by x_{ij} and y_{rj}, respectively. Let the input and output matrices X and Y be

$$X = (x_{ij}) \in R^{m\times n} \quad \text{and} \quad Y = (y_{rj}) \in R^{s\times n}. \quad (1)$$

We assume $X > O$ and $Y > O$. The virtual input and output for DMU_j are defined by

$$V_j = \sum_{i=1}^{m} v_i x_{ij} \quad (j = 1, \ldots, n) \quad (2)$$

and

$$U_j = \sum_{r=1}^{s} u_r y_{rj}, \quad (j = 1, \ldots, n) \quad (3)$$

where (v_i) $((u_r))$ is the the *weight* (or *multiplier*) to the input (output) i (r). Again, we assume $v_i > 0$ $(\forall i)$ and $u_r > 0$ $(\forall r)$.

Now, we observe the ratio of the virtual input *vs.* output:

$$R_j = \frac{U_j}{V_j} = \frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}}. \quad (j = 1, \ldots, n) \quad (4)$$

Since the ratio R_j is invariant under any multiplication by a positive scalar t to $(\boldsymbol{v}, \boldsymbol{u})$, we impose hereafter the simplex constraint to $(\boldsymbol{v}, \boldsymbol{u})$ as follows:

$$\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1. \quad (5)$$

By this constraint, together with the positiveness of multipliers, the feasible $(\boldsymbol{v}, \boldsymbol{u})$ forms the interior of the $(m+s-1)$ dimensional simplex denoted by S. Under the above assumptions, for each $(\bar{\boldsymbol{v}}, \bar{\boldsymbol{u}}) \in S$, there exists at least one DMU_{j_0} that maximizes the ratio R_j $(j = 1, \ldots, n)$ defined by (4). We call DMU_{j_0} *dominates* $(\bar{\boldsymbol{v}}, \bar{\boldsymbol{u}})$. It can be demonstrated that the $(m+s-1)$ dimensional simplex S is divided into a finite number of $(m+s-1)$ dimensional cells dominated by some DMUs. There may exist $(m+s-2)$ or less dimensional dominant DMUs, with the extremal case 0 dimensional (point) dominant DMUs.

3. Assurance Region and Cone-Ratio Models

In applying DEA to actual problems, we should be conscious of the economic/socioeconomic aspect of the problems, which is closely related with the virtual multiplier (weight) $\boldsymbol{v}$ $(\boldsymbol{u})$ to the input (output) items. Although the original DEA models impose no restriction on $\boldsymbol{v}$ and $\boldsymbol{u}$ except positivity (or nonnegativity), we can introduce the relative importance of weights by restricting the feasible region of weights. Along this line, two remarkable models have been proposed, i.e. the assurance region (AR) and the cone-ratio (CR) models.

The AR model imposes lower and upper bounds to the ratio of some selected pairs of weights. For example, we may add a constraint on the ratio of weights to Input 1 and Input 2 as follows:

$$l_{12} \leq \frac{v_2}{v_1} \leq u_{12}, \tag{6}$$

where l_{12} and u_{12} are the lower and upper bounds to the ratio, respectively. Likewise, similar constraints may be added to pairs of some output multipliers and even to multipliers between some input and output multipliers.

On the other hand, in the cone-ratio model, especially in the polyhedral cone-ratio model, some exemplary DMUs will be chosen from among the CCR efficient DMUs as a result of expert knowledge. Then, the optimal weights corresponding to the selected DMUs will be used to form a polyhedral cone for an admissible region of multipliers. However, usually the optimal weights are not uniquely determined. Therefore, we need some other criteria for selecting a reasonable point in the cell. There may be at least three principles for this purpose. The first one, the most restricted case, is to choose the cone as the minimum diameter convex set which makes the exemplary DMUs efficient. The next one, the most relaxed case, is to choose the cone as the convex hull of the exemplary cells. The last one chooses the cone generated by the central points of each exemplary DMUs.

However, it is not easy to implement the above three principles. In fact, the first two might belong to NP-hard problems and the last one depends on the method of choosing the central point for each cell.

4. Practical Methods for Three Cases

Corresponding to the above mentioned general principles, we will propose three practical methods which approximately implement them.

4.1 The Most Restricted Case

Let the chosen exemplary DMUs be DMU_{α_1}, ...,DMU_{α_p}. We solve the following fractional program (FP_k) for each DMU_{α_k} $(k = 1, \ldots, p)$.

$$(\mathrm{FP}_k) \quad \max \quad \frac{\sum_{r=1}^{s} u_r \sum_{j \neq k} y_{r\alpha_j}}{\sum_{i=1}^{m} v_i \sum_{j \neq k} x_{i\alpha_j}}$$

$$\text{subject to} \quad \sum_{i=1}^{m} v_i x_{i\alpha_k} = \sum_{r=1}^{s} u_r y_{r\alpha_k} = 1 \tag{7}$$

$$\sum_{i=1}^{m} v_i x_{i\alpha_j} \geq \sum_{r=1}^{s} u_r y_{r\alpha_j} \quad (\forall j) \tag{8}$$

$$v_i \geq 0 \ (\forall i) \qquad u_r \geq 0 \ (\forall r). \tag{9}$$

The fractional program (FP_k) can be solved as a linear programming problem via the Charnes and Cooper transformation (1962).

4.2 The Most Relaxed Case

This case aims to obtain a cone approximately realizing the convex hull of exemplary cells. Instead of maximizing the objective function in (FP_k), we try to minimize it, subject to the same constraints. Thus the objective is:

$$\text{minimize} \quad \frac{\sum_{r=1}^{s} u_r \sum_{j \neq k} y_{r\alpha_j}}{\sum_{i=1}^{m} v_i \sum_{j \neq k} x_{i\alpha_j}}. \tag{10}$$

This programming will find a vertex in the cell DMU_k which is, in a sense, farthest from other exemplary DMUs. Let the optimal solution be $(\bar{v}_k^*, \bar{u}_k^*)$ $(k = 1, \ldots, p)$, which will be utilized to form the cone for the cone-ratio model.

4.3 The Central Case

There may be several approaches for obtaining relatively interior solutions, among which we will explain two.

4.3.1 Primal-Dual Interior Point Method

The primal-dual interior point methods for linear programming problem will theoretically converge to the center of the optimal facet of the problem and the solution is strictly complementary.

4.3.2 Parametric Linear Programming Approach

If a strictly complementary solution, i.e. $\boldsymbol{v}^* > 0$ and $\boldsymbol{u}^* > 0$, is required instead of the central one, we can obtain one, by a simplex based parametric programming method.

5. Enumeration of Optimal Vertices

It is interesting to know all the vertices $(\boldsymbol{v}, \boldsymbol{u})$ of the convex polyhedron, which makes a DMU efficient, not only for the purposes mentioned in the preceding sections but also for understanding the overall positioning of the efficient DMU in the $(\boldsymbol{v}, \boldsymbol{u})$ space.

Recently, Fukuda (1993) has developed an algorithm and software[1] for enumerating all vertices of a convex polyhedron defined by a system of linear inequalities, base on the Double Description Method (1958). This software works efficiently for medium size problems.

6. An Example

An example will be exhibited at the presentation.

[1] The free software "cdd.c" is available via anonymous ftp from ftp.epfl.ch (directory incoming/dma).