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The "branch-and-support" method for the maximum stable set problem

01306430 電気通信大学 田村 明久 Akihisa TAMURA
02003590 東京工業大学 宇野 毅明 Takeaki UNO*

1 Introduction

The maximum stable set problem (or equivalently, the maximum clique problem), is one of the most well-studied problems in combinatorial optimization. We are given an undirected graph $G = (V, E)$, and the objective is to find a maximum subset of vertices S such that no two elements of S are joined by an edge. Despite this deceptively simple formulation, computationally speaking it remains one of the most difficult problems and has eluded efforts to find an efficient algorithm, in spite of the efforts of many researchers.

In this talk, we propose a new exact strategy which we call the "branch-and-support" method for this problem. Like the branch-and-cut method, it mainly takes a polyhedral approach and only branches when absolutely necessary, but the philosophy behind is different. Basically, the branch-and-cut method starts with a linear program which contains the stable set polytope in its feasible region, and tries to "trim" the extra parts by adding new (facet-inducing) inequalities that cut off fractional optimal solutions. Our branch-and-support method also starts with a linear program, but the difference is that instead of eliminating fractional solutions, it attempts to provide a certificate that a given (good) incumbent is optimal. To do so, it adds only inequalites which "support" the given incumbent, Of course, it also uses the linear relaxation to improve the present incument.

2 Properties of the stable set polytope

Let $G = (V, E)$ be an undirected graph with vertex set V, and edge set E. We consider the family of all stable sets of G. Of course, this family will be composed of subsets of V. For each subset F of V, the *characteristic vector* of F, denoted by $\chi(F)$, is the $0-1$ vector whose v component is 1 if $v \in F$, and 0 otherwise. We define the $0-1$ polytope STAB(G) to be the convex hull of the set of the characteristic vectors of all stable sets of G, i.e.,

$$\mathrm{STAB}(G) = \mathrm{conv}\{\chi(S) \mid S \text{ is stable in } G\}.$$

STAB(G) is called the *stable set polytope* of G.

The stable set polytope is a well studied object, and there are many types of valid inequalities known in the literature [1]. These inequalities perform an essential role in such methods as the branch-and-cut, as well as our new branch-and-support. Perhaps the most well known are the so-called clique inequalities and odd hole inequalities. The *clique inequalities* say that for a given clique K of

G,

$$\sum_{v \in K} x_v \leq 1$$

is valid for STAB(G). Similarly, the *odd hole inequalities* say that for a given hole H whose size is odd,

$$\sum_{v \in H} x_v \leq \frac{|H|-1}{2}$$

is also valid for STAB(G).

Another important property of STAB(G) is that optimizing a linear function over a special type of face, can be easily accomplished. Namely, for any two stable sets S^1 and S^2, the maximum stable set in the graph induced by $S^1 \cup S^2$ and be easily found (this property is generalized in [2]). That is, it suffices to find a maximum matching in the bipartite graph induced by $S^1 \cup S^2$. This property will be used to improve incumbents in our algorithm.

3 The branch-and-support method

In this section, we briefly describe our new method. As we have mentioned, the main idea is to try to provide a certificate that a incumbent which we believe is optimal, is indeed optimal. Basically, it executes the folowing steps.

1. Find an incumbent S^1, construct an LP relaxation and solve it.
2. If the optimal value z^* of the relaxation satisfies $z^* - |S^1| < 1$, then stop; S^1 is a maximum stable set.
3. Otherwise, construct another stable set S^2 (possibly using the optimal solution).
4. Find a maximum stable set in the bipartite graph induced by $S^1 \cup S^2$, and update S^1 if necessary.
5. Add a new inequality which supports S^1, and cuts off the fractional LP solution (if possible), solve the new relaxation and go to 2.

Of course, we cannot always obtain a maximum stable set of G by the above procedure. To guarantee optimality, we may have to branch. However we omit details of the branching procedure, since we employ the natural strategies.

References

[1] Grötschel, M., Lovász, L. and Schrijver, A. (1988), *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag.

[2] Ikebe, Y. T. and Tamura, A. (to appear in), "Ideal polytopes and face structures of some combinatorial optimization problems," *Mathematical Programming.*