1995年度日本オペレーションズ・リサーチ学会
春季研究発表会

A New Approach to Analyze Multiclass M/G/1 Queues

– Steady state mean waiting times –

02991100 University of Tsukuba Sung Jo Hong
01505150 University of Tsukuba Tetsuji Hirayama*

1 Introduction

We introduce a new approach to analyze the multiclass M/G/1 queueing system. In [3], we have defined some cost functions that represent some system performance measures. Then we have derived their values at arbitrary state of the system. In this paper, we will derive thier steady state values.

2 Preliminaries

A single server serves J types of customers. Class i customers arrive at station i from outside the system according to a Poisson process with rate λ_i ($i = 1, \ldots, J$). The overall arrival rate is denoted by $\lambda \equiv \sum_{i=1}^{J} \lambda_i$. Service times S_i of class i customers are arbitrarily distributed. We consider priority scheduling algorithms where station i has a priority over station j if $i < j$. A scheduling discipline for each station is either FCFS or exhaustive. We consider nonpreemptive disciplines only.

Let κ denote the station that is currently selected by the server. The remaining service time of a customer being served currently is denoted by r. Each time interval from when the server selects one of the stations until the first time when the server switchs services to another station is called a *service period.* A set of these periods is denoted by $\Pi \equiv \{0, 1, \ldots, J\}$. Let g denote the number of customers in the system who are scheduled to be served during the current service period (*customers within the gate*). g does not count a customer who are served currently. Hence, at any time epoch, customers in each station are classified into the following three types: (1) a customer being served currently, (2) customers within the gate, and (3) *waiting customers* who are not currently served nor scheduled to serve during the current service period. The number of waiting customers at station i is denoted by n_i. Further let $\mathrm{n} \equiv (n_1, \ldots, n_J)$. The customer list L is used to specify a service order of customers in the system. Let $\mathrm{x}(t) \equiv (\kappa(t), r(t), g(t), \mathrm{n}(t), L(t))$ be a vector composed of these values at time t. The stochastic process $\mathcal{Q} \equiv \{\mathrm{x}(t) = (\kappa(t), r(t), g(t), \mathrm{n}(t), L(t)) : t \geq 0\}$ represents an evolution of the system.

Let $\mathrm{x} = (\kappa, r, g, \mathrm{n}, L)$ be an arbitrary state just prior to an arrival epoch of a tagged class j customer ($j = 1, \ldots, J$). Then the cost function $W_j(\mathrm{x})$ is defined as his mean *waiting time* spent until his service period begins. The cost function $H_j(\mathrm{x}, i)$ is defined as his mean waiting time spent during the system is in service periods of station i until his service period begins ($i = 1, \ldots, J$). The cost function $G_j(\mathrm{x})$ is defined as his mean waiting time spent from when his service period begins until his service is started.

Then by appropriately choosing coefficients, it has been shown in [3] that: for $\kappa \notin \Pi_E$,

$$\begin{aligned} W_j(\mathrm{x}) &= r\varphi_j(\kappa) + g\psi_j(\kappa) + \mathrm{n}\mathcal{W}_j, \\ H_j(\mathrm{x}, i) &= r\varphi_j(\kappa, i) + g\psi_j(\kappa, i) + \mathrm{n}\mathcal{H}_j(i), \\ G_j(\mathrm{x}) &= r\eta_j(\kappa) + g\theta_j(\kappa) + \mathrm{n}\mathcal{G}_j, \end{aligned} \quad (2.1)$$

and for $\kappa \in \Pi_E$,

$$\begin{aligned} W_j(\mathrm{x}) &= r\varphi_j(\kappa) + g\psi_j(\kappa) + \mathrm{n}\mathcal{W}_j + \mathrm{n}\mathcal{W}_j(\kappa), \\ H_j(\mathrm{x}, i) &= r\varphi_j(\kappa, i) + g\psi_j(\kappa, i) + \mathrm{n}\mathcal{H}_j(i) + \mathrm{n}\mathcal{H}_j(\kappa, i), \\ G_j(\mathrm{x}) &= r\eta_j(\kappa) + g\theta_j(\kappa) + \mathrm{n}\mathcal{G}_j + \mathrm{n}\mathcal{G}_j(\kappa), \end{aligned} \quad (2.2)$$

for $i = 1, \ldots, J$.

3 Steady state analysis

In this section, we evaluate steady state values ($\bar{W}_j$, $\bar{H}_j$ and $\bar{G}_j$) of the cost functions W_j, H_j and G_j.

Let σ_j^e be the arrival epoch of the e^{th} arriving customer at station j ($j = 1, \ldots, J$ and $e = 1, 2, \ldots$). Let W_j^e be his waiting time at station j spent until his service period begins. Let $H_j^e(i)$ be his waiting time at station j spent during the system is in service periods of station $i \in \Pi_E$ until his service period begins. Let G_j^e be his waiting time at station j spent from when his service period begins until his service is started. Now let

$$\begin{aligned} \bar{W}_j(\kappa) &\equiv \lim_{N \to \infty} \frac{1}{N} \sum_{e=1}^{N} W_j^e 1\{\kappa(\sigma_j^e) = \kappa\}, \\ \bar{H}_j(\kappa, i) &\equiv \lim_{N \to \infty} \frac{1}{N} \sum_{e=1}^{N} H_j^e(i) 1\{\kappa(\sigma_j^e) = \kappa\}, \\ \bar{G}_j(\kappa) &\equiv \lim_{N \to \infty} \frac{1}{N} \sum_{e=1}^{N} G_j^e 1\{\kappa(\sigma_j^e) = \kappa\}, \end{aligned} \quad (3.3)$$

for $j = 1, \ldots, J, i \in \Pi_E$ and $\kappa \in \Pi$, and where $1\{\mathcal{K}\}$ denotes an indicator of an event $\mathcal{K}$. Further we define $\bar{W}_j \equiv \sum_{\kappa=0}^{J} \bar{W}_j(\kappa)$, $\bar{H}_j(i) \equiv \sum_{\kappa=0}^{J} \bar{H}_j(\kappa, i)$ and $\bar{G}_j \equiv \sum_{\kappa=0}^{J} \bar{G}_j(\kappa)$ for $j = 1, \ldots, J$ and $i \in \Pi_E$.

We assume that the process $\mathcal{Q}$ is regenerative [4]. Let N_B^j be the number of class j customers served during a regenerative cycle ($j = 1, \ldots, J$). We assume that the system is initially empty and that $E[N_B^j] < \infty$. The customer average value $\bar{\mathrm{x}}^{j\kappa} \equiv (\kappa\bar{q}^{j\kappa}, \bar{r}^{j\kappa}, \bar{g}^{j\kappa}, \bar{\mathrm{n}}^{j\kappa}, \bar{L}^{j\kappa})$ of the process just prior to class j arrivals are defined by:

$$\bar{\mathrm{x}}^{j\kappa} \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{e=1}^{N} \mathrm{x}(\sigma_j^e) 1\{\kappa(\sigma_j^e) = \kappa\}, \quad (3.4)$$

for $j = 1, \ldots, J$ and $\kappa \in \Pi$. Then we assume that $E[\sum_{e=1}^{N_B^j} \mathrm{x}(\sigma_j^e) 1\{\kappa(\sigma_j^e) = \kappa\}] < \infty$. Further let the customer average value of the process just prior to class

j arrivals is defined by: $\tilde{\mathrm{x}}^j \equiv (\tilde{\kappa}^j, \tilde{r}^j, \tilde{g}^j, \tilde{\mathrm{n}}^j, \tilde{L}^j) \equiv \sum_{\kappa=0}^{J} \tilde{\mathrm{x}}^{j\kappa}$ $(j = 1, \ldots, J)$. The time average values $\tilde{\mathrm{x}}^\kappa \equiv (\kappa\tilde{q}^\kappa, \tilde{r}^\kappa, \tilde{g}^\kappa, \tilde{\mathrm{n}}^\kappa, \tilde{L}^\kappa)$ and $\tilde{\mathrm{x}} \equiv (\tilde{\kappa}, \tilde{r}, \tilde{g}, \tilde{\mathrm{n}}, \tilde{L})$ of the process is defined by:

$$\tilde{\mathrm{x}}^\kappa \equiv \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{x}(s)1\{\kappa(s) = \kappa\}ds \tag{3.5}$$

for $\kappa \in \Pi$ and $\tilde{\mathrm{x}} \equiv \sum_{\kappa=0}^{J} \tilde{\mathrm{x}}^\kappa = \lim_{t\to\infty} \int_0^t \mathrm{x}(s)ds/t$.

Now we get the following representations concerned with the steady state values of the cost functions. From (2.1), (2.2), (3.3) and (3.4), we can show that: for $\kappa \notin \Pi_E$,

$$\begin{aligned} \bar{W}_j(\kappa) &= \tilde{r}^{j\kappa}\varphi_j(\kappa) + \tilde{g}^{j\kappa}\psi_j(\kappa) + \tilde{\mathrm{n}}^{j\kappa}\mathcal{W}_j, \\ \bar{H}_j(\kappa, i) &= \tilde{r}^{j\kappa}\varphi_j(\kappa, i) + \tilde{g}^{j\kappa}\psi_j(\kappa, i) + \tilde{\mathrm{n}}^{j\kappa}\mathcal{H}_j(i), \\ \bar{G}_j(\kappa) &= \tilde{r}^{j\kappa}\eta_j(\kappa) + \tilde{g}^{j\kappa}\theta_j(\kappa) + \tilde{\mathrm{n}}^{j\kappa}\mathcal{G}_j, \end{aligned} \tag{3.6}$$

and for $\kappa \in \Pi_E$,

$$\begin{aligned} \bar{W}_j(\kappa) &= \tilde{r}^{j\kappa}\varphi_j(\kappa) + \tilde{g}^{j\kappa}\psi_j(\kappa) + \tilde{\mathrm{n}}^{j\kappa}\mathcal{W}_j + \tilde{\mathrm{n}}^{j\kappa}\mathcal{W}_j(\kappa), \\ \bar{H}_j(\kappa, i) &= \tilde{r}^{j\kappa}\varphi_j(\kappa, i) + \tilde{g}^{j\kappa}\psi_j(\kappa, i) \\ &\quad + \tilde{\mathrm{n}}^{j\kappa}\mathcal{H}_j(i) + \tilde{\mathrm{n}}^{j\kappa}\mathcal{H}_j(\kappa, i), \\ \bar{G}_j(\kappa) &= \tilde{r}^{j\kappa}\eta_j(\kappa) + \tilde{g}^{j\kappa}\theta_j(\kappa) + \tilde{\mathrm{n}}^{j\kappa}\mathcal{G}_j + \tilde{\mathrm{n}}^{j\kappa}\mathcal{G}_j(\kappa), \end{aligned} \tag{3.7}$$

for $j = 1, \ldots, J$ and $i \in \Pi_E$.

The steady state value $\tilde{r}^\kappa$ of a remaining service time is given by

$$\tilde{r}^\kappa = \frac{\lambda_\kappa E[S_\kappa^2]}{2}, \quad \kappa = 1, \ldots, J. \tag{3.8}$$

It is obvious that $\tilde{r}^0 = 0$. From the Poisson Arrivals See Time Averages (PASTA) property [5], $\tilde{r}^\kappa$ is equivalent to $\tilde{r}^{j\kappa}$ ($j = 1, \ldots, J$ and $\kappa \in \Pi$). We use the generalized Little's formula ($H = \lambda G$) [1] that equates the time average values of the costs with the customer average values of the costs to obtain

$$\begin{aligned} \tilde{n}_j &= \lambda_j \bar{W}_j, \quad j = 1, \ldots, J, \\ \tilde{n}_j^i &= \lambda_j \bar{H}_j(i), \quad j = 1, \ldots, J \text{ and } i \in \Pi_E, \\ \tilde{g}^j &= \lambda_j \bar{G}_j, \quad j = 1, \ldots, J. \end{aligned} \tag{3.9}$$

Obviously, we have $\tilde{n}_j^0 = 0$ $(j = 1, \ldots, J)$ and $\tilde{g}^0 = 0$.

From these expressions and the PASTA property, we can show

$$\begin{aligned} \tilde{n}_j &= \lambda_j\{\sum_{\kappa=1}^{J} \tilde{r}^\kappa\varphi_j(\kappa) + \sum_{\kappa=1}^{J} \tilde{g}^\kappa\psi_j(\kappa) + \tilde{\mathrm{n}}\mathcal{W}_j + \sum_{\kappa\in\Pi_E} \tilde{\mathrm{n}}^\kappa\mathcal{W}_j(\kappa)\}, \\ \tilde{n}_j^i &= \lambda_j\{\sum_{\kappa=1}^{J} \tilde{r}^\kappa\varphi_j(\kappa, i) + \sum_{\kappa=1}^{J} \tilde{g}^\kappa\psi_j(\kappa, i) \\ &\quad + \tilde{\mathrm{n}}\mathcal{H}_j(i) + \sum_{\kappa\in\Pi_E} \tilde{\mathrm{n}}^\kappa\mathcal{H}_j(\kappa, i)\}, \\ \tilde{g}^j &= \lambda_j\{\sum_{\kappa=1}^{J} \tilde{r}^\kappa\eta_j(\kappa) + \sum_{\kappa=1}^{J} \tilde{g}^\kappa\theta_j(\kappa) + \tilde{\mathrm{n}}\mathcal{G}_j + \sum_{\kappa\in\Pi_E} \tilde{\mathrm{n}}^\kappa\mathcal{G}_j(\kappa)\}, \end{aligned} \tag{3.10}$$

for $j = 1, \ldots, J$ and $i \in \Pi_E$.

Let $\tilde{\mathrm{g}} \equiv (\tilde{g}^1, \ldots, \tilde{g}^J)$ and by appropriately choosing vectors and matrices, we have

$$\begin{aligned} \tilde{\mathrm{n}} &= \left\{\mathrm{s}_w + \tilde{\mathrm{g}}\Psi + \tilde{\mathrm{n}}\mathcal{W} + \sum_{\kappa\in\Pi_E} \tilde{\mathrm{n}}^\kappa\mathcal{W}(\kappa)\right\}\Lambda, \\ \tilde{\mathrm{n}}^i &= \left\{\mathrm{s}_w(i) + \tilde{\mathrm{g}}\Psi(i) + \tilde{\mathrm{n}}\mathcal{H}(i) + \sum_{\kappa\in\Pi_E} \tilde{\mathrm{n}}^\kappa\mathcal{H}(\kappa, i)\right\}\Lambda, \\ \tilde{\mathrm{g}} &= \left\{\mathrm{s}_g + \tilde{\mathrm{g}}\Theta + \tilde{\mathrm{n}}\mathcal{G} + \sum_{\kappa\in\Pi_E} \tilde{\mathrm{n}}^\kappa\mathcal{G}(\kappa)\right\}\Lambda, \end{aligned} \tag{3.11}$$

for $i \in \Pi_E$ where, for example,

$$\begin{aligned} \Lambda &\equiv diag\{\lambda_j : j = 1, \ldots, J\}, \mathcal{W}(\kappa) \equiv (\mathcal{W}_1(\kappa), \ldots, \mathcal{W}_J(\kappa)), \\ \mathcal{W} &\equiv (\mathcal{W}_1, \ldots, \mathcal{W}_J), \quad \mathrm{s}_w \equiv \sum_{\kappa=1}^{J} \tilde{r}^\kappa(\varphi_1(\kappa), \ldots, \varphi_J(\kappa)), \\ \Psi &\equiv \begin{pmatrix} \psi_1(1) & \cdots & \psi_J(1) \\ \vdots & \ddots & \vdots \\ \psi_1(J) & \cdots & \psi_J(J) \end{pmatrix}, \quad \kappa \in \Pi_E. \end{aligned}$$

Let J_E be the number of the stations with exhaustive disciplines and let $i_1, \ldots, i_{J_E}$ denote the stations with exhaustive disciplines. Then $\Pi_E = \{i_1, \ldots, i_{J_E}\}$. Further let $J^* = (2 + J_E) \times J$ be the number of the unknowns $\tilde{\mathrm{g}}, \tilde{\mathrm{n}}, \tilde{\mathrm{n}}^i$ $(i = i_1, \ldots, i_{J_E})$. We define vectors and matrices:

$$\begin{aligned} &\tilde{\mathrm{y}} \equiv (\tilde{\mathrm{g}}, \tilde{\mathrm{n}}, \tilde{\mathrm{n}}^{i_1}, \ldots, \tilde{\mathrm{n}}^{i_{J_E}}) \in R^{1\times J^*}, \\ &\mathrm{s} \equiv (\mathrm{s}_g, \mathrm{s}_w, \mathrm{s}_w(i_1), \ldots, \mathrm{s}_w(i_{J_E})) \in R^{1\times J^*}, \\ &\mathrm{S} \equiv \begin{pmatrix} \Theta & \Psi & \Psi(i_1) & \cdots & \Psi(i_{J_E}) \\ \mathcal{G} & \mathcal{W} & \mathcal{H}(i_1) & \cdots & \mathcal{H}(i_{J_E}) \\ \mathcal{G}(i_1) & \mathcal{W}(i_1) & \mathcal{H}(i_1, i_1) & \cdots & \mathcal{H}(i_1, i_{J_E}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{G}(i_{J_E}) & \mathcal{W}(i_{J_E}) & \mathcal{H}(i_{J_E}, i_1) & \cdots & \mathcal{H}(i_{J_E}, i_{J_E}) \end{pmatrix}, \\ &\Lambda_* \equiv diag\{\lambda_1, \ldots, \lambda_J, \lambda_1, \ldots, \lambda_J, \ldots, \lambda_1, \ldots, \lambda_J\}, \end{aligned}$$

where $\mathrm{S}, \Lambda_* \in R^{J^*\times J^*}$. Then we arrive at an equation that determines the steady state value of the components of the process:

$$\tilde{\mathrm{y}} = \{\mathrm{s} + \tilde{\mathrm{y}}\mathrm{S}\}\Lambda_*. \tag{3.12}$$

If we assume that the inverse matrices exist, we have

$$\tilde{\mathrm{y}} = \mathrm{s}\left(\Lambda_*^{-1} - \mathrm{S}\right)^{-1}. \tag{3.13}$$

Finally, we can get steady state values $\tilde{\mathrm{n}}_s$ of the number of customers in the system and steady state values $\bar{\mathrm{w}}_s = (\bar{W}_1 + \bar{G}_1, \ldots, \bar{W}_J + \bar{G}_J)$ of the mean waiting times:

$$\tilde{\mathrm{n}}_s = \tilde{\mathrm{n}} + \tilde{\mathrm{g}}, \tag{3.14}$$

$$\bar{\mathrm{w}}_s = (\tilde{\mathrm{n}} + \tilde{\mathrm{g}})\Lambda^{-1}. \tag{3.15}$$

References

[1] P. Glynn & W. Whitt, Oper. Res. 37 (1989) 634-644.

[2] S.J. Hong, et al., Asian-Pacific J. of Oper. Res. 10 (1993) 233-249.

[3] S.J. Hong & T. Hirayama, A new approach to analyze multiclass M/G/1 queues (I) · (II), in proceedings of fall conf. of JORS, 1994.

[4] S.M. Ross, *Applied Probability Models with Optimization Applications*, (Holden-Day, 1976).

[5] R.W. Wolff, Oper. Res. 30 (1982) 223-231.