1995年度日本オペレーションズ・リサーチ学会
春季研究発表会

On the maximum balanced k-flow problem

01403552 Fukushima Fukushima Akira NAKAYAMA

1. Introduction

D.K. Wagner and H. Wan considered the *maximum k-flow problem* of finding a maximum k-flow determining how much arc capacity to purchase for each arc and how much flow to send so as to maximize the net profit, where the capacity of any arc can be increased at a per-unit cost and every unit of flow sent from the source to the sink yields a payoff k. Here, we consider the *maximum balanced k-flow problem* of finding a maximum k-flow in a network with an additional constraint described in terms of a balancing rate function. We show that the maximum balanced k-flow problem can be regarded as a generalization of the maximum balanced flow problem, provided that all of the input data are rational.

2. Maximum balanced k-flows

We denote the sets of reals and of nonnegative reals by $\mathbf{R}$ and $\mathbf{R}_+$. Let $G = (V, A)$ be a directed graph with a vertex set V and an arc set A. We use $x(T) \equiv \sum_{t \in T} x(t)$ for a function x on A and $T \subseteq A$. Consider $N = (G = (V, A), u; s^+, s^-)$, where $u : A \to \mathbf{R}_+ - \{0\}$ is capacity function on A, and s^+ and s^- ($s^+ \neq s^-$) are the source and the sink of N. For a function $f : A \to \mathbf{R}$, define the *boundary* $\partial f : V \to \mathbf{R}$ of f by

$$\partial f(v) \equiv \textstyle\sum_{a \in \delta^+ v} f(a) - \sum_{a \in \delta^- v} f(a), \quad v \in V,$$

where $\delta^+ v$ ($\delta^- v$) is the set of arcs in G which have v as their initial (terminal) vertices. Given N, the *maximum flow problem* (MF) is as follows:

(MF): Maximize $\mathrm{val}_N(f)$ s.t.

$$\partial f(v) = 0, \qquad v \in V - \{s^+, s^-\}, \qquad (1)$$

$$0 \leq f(a) \leq u(a), \qquad a \in A, \qquad (2)$$

where f satisfying (1) is a *flow* in N of *flow value* $\mathrm{val}_N(f) \equiv \partial f(s^+)$. A flow f is *feasible* if f satisfies (2). A feasible flow f maximizing $\mathrm{val}_N(f)$ is a *maximum* flow in N. For $S \subseteq V$ such that $s^+ \in S$ and $s^- \in V - S$, $K(S) \equiv \{(u, v) \in A : u \in S, v \in V - S\}$ is a *cut* of G. The *value* of a cut $K(S)$ in N is $v(K(S)) \equiv \sum_{a \in K(S)} u(a)$. A *minimum* cut is a cut minimizing the value. Given a *balancing rate* function $\alpha : A \to \mathbf{R}_+ - \{0\}$, the *maximum balanced flow problem* (MBF) in $N_1 = (G = (V, A), u, \alpha; s^+, s^-)$ is:

(MBF): Maximize $\mathrm{val}_{N_1}(f)$ s.t. (1), (2) and

$$f(a) \leq \alpha(a)\mathrm{val}_{N_1}(f), \quad a \in A. \qquad (3)$$

A flow f satisfying (3) is *balanced* in N_1. A feasible balanced flow f maximizing $\mathrm{val}_{N_1}(f)$ is a *maximum* balanced flow in N_1. Given integral *cost* function $c : A \to \mathbf{R}_+$ and an integer $k > 0$, Wanger and Wan ([1]) discussed the *maximum k-flow problem* $(\mathrm{MF})^c_k$ in $N_2 = (G = (V, A), u, c, k; s^+, s^-)$ defined as follows:

$(\mathrm{MF})^c_k$: Maximize $\mathrm{val}_{N_2}(k, c, f) \equiv k\mathrm{val}_{N_2}(f) - \sum_{a \in A} c(a) \max\{f(a) - u(a), 0\}$ s.t. (1), A flow f is a *k-flow* in N_2. A k-flow f maximizing $\mathrm{val}_{N_2}(k,c,f)$ is a *maximum* k-flow in N_2. Given $N_3 = (G = (V,A), u, \alpha, c, k; s^+, s^-)$, consider the *maximum balanced k-flow problem* $(\mathrm{MBF})_k^c$ defined as follows:

$(\mathrm{MBF})_k^c$: Maximize $\mathrm{val}_{N_3}(k,c,f)$ s.t. (1),
$$0 \le f(a) \le \alpha(a)\mathrm{val}_{N_3}(k,c,f), \quad a \in A, \qquad (4)$$

A flow f satisfying (4) is a *balanced k-flow* in N_3. A balanced k-flow f maximizing $\mathrm{val}_{N_3}(k,c,f)$ is a *maximum* balanced k-flow in N_3. We introduce a parametric problem $(\mathrm{MF})_y$ of $N_y = (G = (V,A), \alpha, u; s^+, s^-, y)$ with parameter y.

$(\mathrm{MF})_y$: Maximize $\mathrm{val}_{N_y}(f)$ s.t. (1) and
$$0 \le f(a) \le \min\{u(a), \alpha(a)y\}, \quad a \in A. \qquad (5)$$

Let $K(S_{y'})$ be a minimum cut in $N_{y'}$ and $K^1(S_{y'}) = \{a \in K(S_{y'}) : u(a) \le \alpha(a)y'\}$ for a fixed value $y = y'$. Let $y^* = \max\{y' : y' = \mathrm{val}_{N_{y'}}(f_{y'})\}$, where $f_{y'}3e$ is a maximum flow in $N_{y'}$. Then we have

$$y^* = u(K^1(S_{y^*}))/(1-\alpha(K^2(S_{y^*})), \qquad (6)$$

where $K^2(S_{y^*}) = K(S_{y^*}) - K^1(S_{y^*})$.

3. Analysis on rational data

We show that problem $(\mathrm{MBF})_1^c$ can be regarded as a generalization of problem (MBF), provided that all of the given data are rational. We assume that u is integral and $\alpha(a) = \zeta(a)/\eta(a)$ ($a \in A$) for some positive integers $\zeta(a)$ and $\eta(a)$. Let $u_\epsilon(a) \equiv u(a) + \epsilon(a)$ for given nonnegative real numbers $\epsilon(a)$ $(a \in A)$. Then the optimal value y_ϵ^* (y^*) in $N_y^\epsilon = (G, u_\epsilon, \alpha; s^+, s^-, y)$ (N_y) is given by

$$y_\epsilon^* = u_\epsilon(K^1(S_{y_\epsilon^*}))/(1-\alpha(K^2(S_{y_\epsilon^*}))) , \qquad (7)$$

$$y^* = u(K^1(S_{y^*}))/(1-\alpha(K^2(S_{y^*}))) , \qquad (8)$$

where $K(S_{y_\epsilon^*})$ ($K(S_{y^*})$) is a minimum cut in $N_{y_\epsilon^*}^\epsilon$ (N_{y^*}). Note that y_ϵ^* (y^*) is the flow value of a maximum balanced flow in $N_1^\epsilon = (G = (V,A), u_\epsilon, \alpha; s^+, s^-)$ (N_1). Let $\Gamma \equiv \Pi_{a\in A}\eta(a)$, and the following two lemmas holds.

Lemma 1: *Let $c(a) > C_1 \equiv |A|\Gamma$ $(a \in A)$ and f_ϵ^* be a maximum balanced flow in N_1^ϵ. Then we have $y^* > \mathrm{val}_{N_3}(1,c,f_\epsilon^*)$ if*
$$y^* \ge u(K^1(S_{y_\epsilon^*}))/(1-\alpha(K^2(S_{y_\epsilon^*}))).$$ ▩

Lemma 2: *Let $u_{\max} = \max_{a\in A} u(a)$ and $c(a) > C_2 \equiv C_1(C_1\Gamma u_{\max}+1)$ $(a \in A)$. Then we have $y^* > \mathrm{val}_{N_3}(1,c,f_\epsilon^*)$ if*
$$y^* < u(K^1(S_{y_\epsilon^*}))/(1-\alpha(K^2(S_{y_\epsilon^*}))).$$ ▩

Consider network N_3, where $c(a) > C_2$ for each $a \in A$. Let f_1 be any balanced 1-flow in N_3. From $c > 0$ and (4) we have $f_1(a) \le \alpha(a)\mathrm{val}_{N_3}(f_1)$, which implies that f_1 is a balanced flow in N_1. Define ϵ by

$$\epsilon(a) = \begin{cases} 0 & (f_1(a) \le u(a)\ ), \\ f_1(a) - u(a) & (\text{otherwise}). \end{cases}$$

Then f_1 is a feasible balanced flow in N_1^ϵ. From this, we have $\mathrm{val}_{N_3}(1,c,f_1) \le \mathrm{val}_{N_1^\epsilon}(f_\epsilon^*)$. From $\mathrm{val}_{N_3}(f_1) \le \mathrm{val}_{N_3}(f_\epsilon^*)$, we have $\mathrm{val}_{N_3}(1,c,f_1) \le \mathrm{val}_{N_3}(1,c,f_\epsilon^*)$. From lemmas 1 and 2, we have $\mathrm{val}_{N_3}(1,c,f_1) < y^*$. Hence we have the following theorem.

Theorem 3: *The maximum balanced 1-flow problem can be regarded as a generalization of the maximum balanced flow problem.* ▩

4. References

[1] D.K. Wagner and H. Wan: A polynomial-time simplex method for the maximum k-flow problem, *Mathematical Programming*, Vol.60, No.1 (1993), 115-123.