1995年度日本オペレーションズ・リサーチ学会
春季研究発表会

Experimental analysis of a semidefinite programming approach to the graph partitioning problem

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1 Introduction

There has been much interest in semidefinite programming (SDP). The objective of our study is to analyze the power and the applicability of SDP approaches to combinatorial optimization via extensive computational experiments. The SDP is important not only because it is solvable in polynomial time by an interior point or ellipsoid algorithm (see, for example, Kojima's tutorial paper in this issue) but also because it leads to tighter relaxations than classical linear programming relaxations.

Recently, Goemans and Williamson [2] derived an approximation algorithm for MAX CUT using the SDP relaxation whose worst case ratio is much better than the previously known bound. We apply the same technique to the following problem.

(Graph Partitioning Problem: GPP)
Given a complete undirected graph $G=(V,E)$. Associated with an edge $(i,j)\in E$, there exists a nonnegative weightc_{ij}. Assume that $|V|$ is an even number and let $n=|V|$. A partition of V is a pair (L,R) of vertex sets such that $L\cap R=\emptyset$ and $L\cup R=V$. A partition (L,R) of V is called *uniform* when $|L|=|R|=n/2$. Then the objective of the problem is to find a uniform partition (L,R) of V that minimizes $c(L,R)=\sum_{i\in L,j\in R}c_{ij}$.

The GPP used as a test bed to investigate the power of metaheuristics [1, 3]. Johnson et al. [3] provided benchmark instances on which some heuristics have been compared via extensive computatiional experiments; they showed Kernighan and Lin's heuristic (KL) [4] works well on geometric (structured) problems but the simulated annealing (SA) gives better solutions than KL on random graphs if much computational time is allowed. The authors [1] showed that a variant of tabu search that we call the life span method (LSM: see, for example, Kubo's tutorial paper in this issue) beats both SA and KL on random and geometric graphs.

The organization of the rest of the paper is as follows. Section 2 gives two SDP formulations of the GPP. Based on a SDP relaxation, we derive an approximate algorithm for the GPP in Section 3. In Section 4, we compare the algorithm derived in Section 3 with the LSM via computational experiments. Final section gives some open problems.

2 Formulations

Let $y=(y_{ij})\in\Re^{n(n-1)/2}$ be the edge incidence vector of the uniform partition defined by

$$y_{ij}=\begin{cases}1 & i<j, i\in L, j\in R \text{ or } i\in R, j\in L\\ 0 & \text{otherwise.}\end{cases}\qquad(1)$$

Then the edge formulation of the GPP is

$$\begin{array}{lll}\min & \sum_{i<j}c_{ij}y_{ij} & \\ \text{s.t.} & \sum_{i<j}y_{ij}=n^2/4 & (2)\\ & 4\sum_{i<j}a_ia_jy_{ij}\le(\sum_i a_i)^2 \quad \text{for } a\in\Re^n & (3)\\ & y_{ij}\in\{0,1\} \quad \text{for } i<j. & (4)\end{array}$$

The inequalities (3) can be derived by the inequality $4(\sum_{i\in L}a_i)(\sum_{i\in R}a_i)\le(\sum_{i\in V}a_i)^2$ for any partition (L,R) of V, and be rewritten by the semidefinite constraint

$$\frac{1}{2}J-Y\succeq 0 \qquad (5)$$

where J denotes the all 1's matrix and Y denotes the $n\times n$ symmetric matrix with zero diagonal and ij and ji entries y_{ij}.

By replacing the constraints (4) with $0\le y_{ij}\le 1$ for $i<j$, we derive a semidefinite programming relaxation of the GPP that can be further strengthened by adding the linear inequalities

$$y_{ij}+y_{ik}+y_{jk}\le 2 \quad \text{for } i,j,k \qquad (6)$$

and

$$y_{ij}-y_{ik}-y_{jk}\le 0 \quad \text{for } i,j,k. \qquad (7)$$

We denote this relaxtion by (P).

Let $x=(x_i)\in\Re^n$ be the vertex incidence vector of the uniform partition defined by

$$x_i=\begin{cases}1 & i\in L\\ -1 & i\in R.\end{cases}\qquad(8)$$

Then the vertex formulation of the GPP is

$$\begin{array}{lll}\min & \frac{1}{2}\sum_{i<j}c_{ij}(1-x_ix_j) & \\ \text{s.t.} & \sum_i x_i=0 & (9)\\ & x_i\in\{-1,1\} \quad \text{for } i. & (10)\end{array}$$

Note that for any incidence vectors x and y, $x_ix_j/2=1/2-y_{ij}$ holds.

By replacing one dimensional vector x_i with n-dimensional vector v_i of unit norm, we derive the relaxtion

$$\begin{array}{lll}\min & \frac{1}{2}\sum_{i<j}c_{ij}(1-v_i\cdot v_j) & \\ \text{s.t.} & \sum_i v_i=0 & (11)\\ & \|v_i\|=1 \quad \text{for } i & (12)\end{array}$$

where $v_i\cdot v_j$ represents the inner (dot) product.

By replacing $v_i \cdot v_j$ with ξ_{ij}, we derive another semidefinite relaxation of the GPP

$$\begin{array}{lll} \min & \frac{1}{2}\sum_{i<j} c_{ij}(1-\xi_{ij}) & \\ \text{s.t.} & \sum_i \xi_{ij} = -n/2 & (13) \\ & \Xi \succeq 0 & (14) \\ & \xi_{ii} = 1 \quad \text{for } i. & (15) \end{array}$$

Note that (13) and (14) correspond to (2) and (5), respectively, and a symmetric positive semidefinite matrix $\Xi = (\xi_{ij})$ with $\xi_{ii} = 1$ corresponds to a set of vectors $v_i, i \in V$ satisfying (12). Note also that the dual of the above SDP leads to the (min-max) eigenvalue bound [5].

3 The randomized algorithm

Using the SDP relaxation, we derive the following randomized approximate algorithm for the GPP.

Step 1: Solve (P). Let the optimal solution be $Y = (y_{ij})$.

Step 2: Using a Cholesky decomposition, find a matrix V such that $V^T V = (\frac{1}{2}J - Y)$.

Step 3: Choose a hyperplane through the origin. If the column vector v_i of V lies above the plane, then vertex i is in L; otherwise vertex i is in R.

Remark 1 (P) can be solved by a cutting-plane method. The identification problem of the semidefinite constraints (5) is reduced to finding the minimum eigenvalue of $\frac{1}{2}J - Y$; if the eigenvalue is negative, the corresponding eigenvector a induces a cut (3).

Remark 2 Since the vector v obtained by the above algorithm always satisfies $\sum_{i \in V} v_i = 0$, any hyperplane through the origin derives a uniform partition; so repeated trials of the randomized algorithm may give better solutions.

Remark 3 Using the tighter inequalities than (3) such as gap inequalities or hypermetric inequalities [5], we can derive tighter relaxations that may lead to better approximate solutions. But, unfortunately, the identification of the gap inequality is NP-hard and some heuristic identification scheme will be needed.

Remark 4 For the GPP, the direct application of the technique by Goemans and Williamson does not give a reasonable worse case bound. A critical point is that MAX CUT is the problem of maximizing nonnegative weights, while the GPP is the problem of minimizing non-negative weights. The similar phenomenon occures when we derive the worst case (probabilistic) bounds via the randomized rounding technique of linear progrramming relaxations [6].

4 Computational experiments

We test the randomized algorithm derived in Section 3 on the standard test problems available from DIMACS (ftp.dimacs.rutgers.edu in the directory /DSJ) and TSPLIB problems, and then compare it with the LSM. Developing the LSM for MAX CUT and then comparing with Goemans and Williamson's algorithm would be of interest. Results of computational experiments will be reported in talk.

5 Conclusion

We demonstrate the applicability of the semidefinite programming to the graph partitioning problem. It will be of value to apply the similar technique to the stable set problem and the quadratic assignment problem because both problems have tight relaxations that are natually formulated as semidefinite programming. We can also derive the following (semidefinite) cut for the n-city asymmetric traveling salesman problem ($n \geq 3$):

$$4\sum_{i,j} b_i b_j x_{ij} \leq (\sum_i b_i)^2 \quad \text{for } b \in \Re^n_+$$

$$\sum_{i,j \neq 1} a_i a_j x_{ij} - \sum_i a_i a_1 x_{i1} \leq \cos\frac{\pi}{n}(\sum_i a_i)^2 \quad \text{for } a \in \Re^n.$$

Analyzing the strength of these constraints is an interesting open problem.

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