1996年度日本オペレーションズ・リサーチ学会
春季研究発表会

Convergence Analysis of Some Interior-Point Methods for the Monotone Semidefinite Linear Complementarity Problem

Masakazu KOJIMA Tokyo Institute of Technology
Masayuki SHIDA Kanagawa University
Susumu SHINDOH* The National Defense Academy

1. Introduction.

Let $\mathcal{S}$ denote the set of all $n \times n$ symmetric real matrices. We regard $\mathcal{S}$ an $n(n+1)/2$-dimensional linear space with the inner product $\boldsymbol{X} \bullet \boldsymbol{Y} = \mathrm{Tr}\,\boldsymbol{XY}$ of $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathcal{S}$ and the Frobenius norm $\|\boldsymbol{X}\|_F = (\boldsymbol{X} \bullet \boldsymbol{X})^{1/2}$ of $\boldsymbol{X} \in \mathcal{S}$. We write $\boldsymbol{X} \succ \boldsymbol{O}$ if $\boldsymbol{X} \in \mathcal{S}$ is positive definite, and $\boldsymbol{X} \succeq \boldsymbol{O}$ if $\boldsymbol{X} \in \mathcal{S}$ is positive semidefinite. Here $\boldsymbol{O}$ denotes the $n \times n$ zero matrix. We also use the symbol $\mathcal{S}_+$ for the set of positive semidefinite symmetric matrices.

Let $\mathcal{F}$ be an $n(n+1)/2$-dimensional affine subspace of $\mathcal{S} \times \mathcal{S}$, and

$$\mathcal{F}_+ = \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} \ : \ \boldsymbol{X} \succeq \boldsymbol{O}, \ \boldsymbol{Y} \succeq \boldsymbol{O}\}.$$

We are concerned with the SDLCP (semidefinite linear complementarity problem):

$$\text{Find an } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}_+ \ \text{ such that } \boldsymbol{X} \bullet \boldsymbol{Y} = 0. \tag{1}$$

Let

$$\mathcal{F}_0 = \{(\boldsymbol{X}' - \boldsymbol{X}, \boldsymbol{Y}' - \boldsymbol{Y}) : (\boldsymbol{X}', \boldsymbol{Y}'), (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}\}.$$

Throughout the paper we assume the monotonicity

$$\boldsymbol{U} \bullet \boldsymbol{V} \geq 0 \ \text{ for every } (\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{F}_0. \tag{2}$$

The purpose of the paper is to propose a globally convergent predictor-corrector infeasible-interior-point algorithm, with the use of the Alizadeh-Haeberly-Overton search direction, for the monotone SDLCP and demonstrate its quadratic convergence under the strict complementarity and the nondegeneracy conditions. (See [2] for details. See also [1].)

2. Predictor-Corrector Interior-Point Algorithm.

Let $\zeta \geq 1/n$ be a fixed number. For each $\gamma \in [0, 1]$ and each $\tau \geq 0$, define $\widetilde{\mathcal{N}}(\gamma, \tau) = \left\{ (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+^2 \ : \ \begin{array}{l} (\boldsymbol{XY} + \boldsymbol{YX})/2 \succeq (1-\gamma)\tau \boldsymbol{I}, \\ \boldsymbol{X} \bullet \boldsymbol{Y}/n \leq (1 + \zeta\gamma)\tau \end{array} \right\}$.

Before we run Algorithm 2..2, we build up the hypothesis below.

Hypothesis 2..1. Let $\omega^* \geq 1$. There exists a solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ of the SDLCP (1) such that

$$\omega^* \boldsymbol{X}^0 \succeq \boldsymbol{X}^* \ \text{ and } \omega^* \boldsymbol{Y}^0 \succeq \boldsymbol{Y}^*. \tag{3}$$

Algorithm 2..2.

Step 0: Choose a parameter $\epsilon \geq 0$, a neighborhood parameter $\gamma \in (0, 1)$ and an initial point $(\boldsymbol{X}^0, \boldsymbol{Y}^0) = (\sqrt{\mu^0}\boldsymbol{I}, \sqrt{\mu^0}\boldsymbol{I})$ with some $\mu^0 > 0$. Let $\theta^0 = 1$, $\sigma = 2\omega^*/(1-\gamma) + 1$, $\gamma^0 = 0$ and $k = 0$.

Step 1: If the inequality

$$\theta^k(\boldsymbol{X}^0 \bullet \boldsymbol{Y}^k + \boldsymbol{X}^k \bullet \boldsymbol{Y}^0) \leq \sigma \boldsymbol{X}^k \bullet \boldsymbol{Y}^k \tag{4}$$

does not hold then stop.

Step 2: (Predictor Step) Compute a solution $(d\boldsymbol{X}_p^k, d\boldsymbol{Y}_p^k)$ of the system of equations

$$\left. \begin{array}{l} \boldsymbol{X}^k d\boldsymbol{Y}_p^k + d\boldsymbol{Y}_p^k \boldsymbol{X}^k + d\boldsymbol{X}_p^k \boldsymbol{Y}^k + \boldsymbol{Y}^k d\boldsymbol{X}_p^k \\ = -\boldsymbol{X}^k \boldsymbol{Y}^k - \boldsymbol{Y}^k \boldsymbol{X}^k, \\ (\boldsymbol{X}^k + d\boldsymbol{X}_p^k, \boldsymbol{Y}^k + d\boldsymbol{Y}_p^k) \in \mathcal{F}. \end{array} \right\} \tag{5}$$

Let

$$\left.\begin{array}{rcl} \delta_p^k & = & \dfrac{\|d\boldsymbol{X}_p^k\|_F\|d\boldsymbol{Y}_p^k\|_F}{\theta^k\mu^0}, \\ \hat{\alpha}_p^k & = & \dfrac{2}{\sqrt{1+4\delta_p^k/(\gamma-\gamma^k)}+1}. \end{array}\right\} \quad (6)$$

Choose a step length $\alpha_p^k \in [\hat{\alpha}_p^k, 0]$. Let

$$(\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k) = (\boldsymbol{X}^k, \boldsymbol{Y}^k) + \alpha_p^k(d\boldsymbol{X}_p^k, d\boldsymbol{Y}_p^k)$$
$$\text{and} \quad \theta^{k+1} = (1-\alpha_p^k)\theta^k.$$

Step 3: If $\theta^{k+1} \leq \epsilon$ then stop. If the inequality

$$\theta^{k+1}(\boldsymbol{X}_c^0 \bullet \boldsymbol{Y}_c^k + \boldsymbol{X}_c^k \bullet \boldsymbol{Y}^0) \leq \sigma \boldsymbol{X}_c^k \bullet \boldsymbol{Y}_c^k \quad (7)$$

does not hold then stop.

Step 4: (Corrector Step) Compute a solution $(d\boldsymbol{X}_c^k, d\boldsymbol{Y}_c^k)$ of the system of equations

$$\left.\begin{array}{l} \boldsymbol{X}_c^k d\boldsymbol{Y}_c^k + d\boldsymbol{Y}_c^k \boldsymbol{X}_c^k + d\boldsymbol{X}_c^k \boldsymbol{Y}_c^k + \boldsymbol{Y}_c^k d\boldsymbol{X}_c^k \\ = 2\theta^{k+1}\mu^0 \boldsymbol{I} - \boldsymbol{X}_c^k \boldsymbol{Y}_c^k - \boldsymbol{Y}_c^k \boldsymbol{X}_c^k, \\ (d\boldsymbol{X}_c^k, d\boldsymbol{Y}_c^k) \in \mathcal{F}_0. \end{array}\right\} \quad (8)$$

Let

$$\begin{array}{rcl} \delta_c^k & = & \dfrac{\|d\boldsymbol{X}_c^k\|_F\|d\boldsymbol{Y}_c^k\|_F}{\theta^{k+1}\mu^0}, \\ \hat{\alpha}_c^k & = & \begin{cases} \gamma/(2\delta_c^k) & \text{if } \gamma \leq 2\delta_c^k, \\ 1 & \text{if } \gamma > 2\delta_c^k, \end{cases} \\ \hat{\gamma}^{k+1} & = & \begin{cases} \gamma(1-\gamma/(4\delta_c^k)) & \text{if } \gamma \leq 2\delta_c^k, \\ \delta_c^k & \text{if } \gamma > 2\delta_c^k. \end{cases} \end{array} \quad (9)$$

Choose a step length $\alpha_c^k \in [0,1]$ and γ^{k+1} such that $\gamma^{k+1} \leq \hat{\gamma}^{k+1}$ and $(\boldsymbol{X}_c^k + \alpha_c^k d\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k + \alpha_c^k d\boldsymbol{Y}_c^k) \in \widetilde{\mathcal{N}}(\gamma^{k+1}, \theta^{k+1}\mu^0)$. Let

$$(\boldsymbol{X}^{k+1}, \boldsymbol{Y}^{k+1}) = (\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k) + \alpha_c^k(d\boldsymbol{X}_c^k, d\boldsymbol{Y}_c^k).$$

Step 5: Replace k by $k+1$. Go to Step 1.

3. Local Convergence.

Throughout this section, we assume Hypothesis 2..1 and

Condition 3..1.

1. *(Strict Complementarity)* $\boldsymbol{X}^* + \boldsymbol{Y}^* \succ \boldsymbol{O}$.
2. *(Nondegeneracy)* $(\boldsymbol{U}, \boldsymbol{V}) = (\boldsymbol{O}, \boldsymbol{O})$ *if* $\boldsymbol{X}^*\boldsymbol{V} + \boldsymbol{U}\boldsymbol{Y}^* = \boldsymbol{O}$ *and* $(\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{F}_0$.

Under these assumptions, the solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ of the SDLCP (1) ensured by Hypothesis 2..1 is the unique one.

Assuming that the sequence is infinite, we establish:

Theorem 3..2. *(Local Convergence Theorem) Assume that Hypothesis 2..1 and Condition 3..1 hold. Let* $\{(\boldsymbol{X}^k, \boldsymbol{Y}^k, \boldsymbol{X}_c^k, \boldsymbol{Y}_c^k, \theta^k, \gamma^k)\}$ *be the sequence generated by Algorithm 2..2 with taking* $\epsilon = 0$ *at Step 0.*

1. *The* $\hat{\alpha}_c^k$ *defined in Step 4 satisfies that* $\hat{\alpha}_c^k = 1$ *for every sufficiently large* k.
2. *The* $\hat{\gamma}^{k+1}$ *defined in Step 4 satisfies that* $\hat{\gamma}^{k+1} \to 0$ *as* $k \to \infty$.
3. *The* $\hat{\alpha}_p^k$ *defined in (6) satisfies that* $\hat{\alpha}_p^k \to 1$ *as* $k \to \infty$.
4. *There is a positive constant* η *such that* $\theta^{k+1} \leq \eta(\theta^k)^2$ *for every* $k = 0, 1, 2, \ldots$.

We will also talk about the local superlinear convergence of an another predictor-corrector algorithm. (See [1].)

References

[1] M. Kojima, M. Shida and S. Shindoh, December 1995.

[2] M. Kojima, M. Shida and S. Shindoh, January 1996.