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Dual Axiomatization of the Core

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1. Dualization and Reduction Operators

Let (N,v) be a *cooperative n-person game in characteristic function form*, where $N = \{1,2,3,...,n\}$ $(n \geq 1)$ is the set of *players* and v is a *characteristic function* from 2^N to $\mathcal{R}$ with $v(\emptyset) = 0$. A *payoff vector* of a game (N,v) is a function $x : N \mapsto \mathcal{R}$, and $\mathcal{R}^N$ is the set of all payoff vectors of (N,v). For $x \in \mathcal{R}^N$ and $S \subseteq N$, we denote $x(S) = \sum_{i \in S} x_i$, and $x(\emptyset) = 0$. Let Γ^A be the set of all games (N,v) and $\Gamma \subseteq \Gamma^A$. A *solution* on Γ is a function Φ which associates with each game $(N,v) \in \Gamma$ a subset $\Phi(N,v)$ of $\mathcal{R}^N$. The *core* of (N,v) is defined by $C(N,v) = \{x \in X(N,v) | x(S) \geq v(S) \text{ for all } S \subset N\}$. Here the *pre-imputation set* for (N,v) is defined by $X(N,v) = \{x \in \mathcal{R}^N | x(N) = v(N)\}$. The *anti-core* of (N,v) is defined by $AC(N,v) = \{x \in X(N,v) | x(S) \leq v(S) \text{ for all } S \subset N\}$.

The *dualization operator* $\mathbf{D}$ is an operator of a game $(N,v) \in \Gamma$ which associates with each game (N,v) a *dual* game $\mathbf{D}(N,v) = (N, \mathbf{D}v)$, where $\mathbf{D}v$ is given by $(\mathbf{D}v)(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$. The *dual solution* $\mathbf{D}\Phi(N,v)$ on $\mathbf{D}\Gamma$ of a solution $\Phi(N,v)$ on Γ is defined by $\mathbf{D}\Phi(N,v) = \Phi(N, \mathbf{D}v)$ for any $(N,v) \in \mathbf{D}\Gamma$.

Definition 1 *Let $x \in \mathcal{R}$ and $j \in \mathcal{N}$, where $\mathcal{N}$ is the set of the natural numbers. The* reduction operator $\mathbf{R}^{x,j}$ *is an operator of a game $(N,v) \in \Gamma$ and a characteristic function v which associates with each game an $(n-1)$-person game $\mathbf{R}^{x,j}(N,v) = (N \setminus \{j\}, \mathbf{R}^{x,j}v)$ when $j \in N$ and $n \geq 2$, or the original n-person game (N,v) when $j \notin N$ or $n = 1$, where the characteristic function $\mathbf{R}^{x,j}v$ satisfies, if $j \in N$ and $n \geq 2$, $(\mathbf{R}^{x,j}v)(\emptyset) = 0$ $(\mathbf{R}^{x,j}v)(N \setminus \{j\}) = v(N) - x$. We call $(N \setminus \{j\}, \mathbf{R}^{x,j}v)$ a* reduced game *of (N,v) when $j \in N$ and $n \geq 2$.*

2. Axioms and Dual Axioms

We extend the domain Γ of Φ to Γ^A by $\Phi(N,v) = \emptyset$ for any $(N,v) \in \Gamma^A \setminus \Gamma$. A propositional function p on $\{((N,v),Q) \mid (N,v) \in \Gamma^A, Q \subseteq \mathcal{R}^N\}$ to $\{0,1\}$ is introduced. We use $\mathcal{P}$ as the set of all propositional functions. For a class of games Γ, we consider an equivalence relation $\sim$ on $\mathcal{P}$: $p \sim \tilde{p} \iff p((N,v),Q) = \tilde{p}((N,v),Q)$ for all $(N,v) \in \Gamma$, any $Q \subseteq \mathcal{R}^N$. When we consider a solution Φ on Γ and we put $\Phi(N,v)$ into Q, we have a new equivalence relation $\sim_\Phi$ on $\mathcal{P}$: $p \sim_\Phi \tilde{p} \iff p((N,v),\Phi(N,v)) = \tilde{p}((N,v),\Phi(N,v))$ for all $(N,v) \in \Gamma$. This equivalence relation $\sim_\Phi$ is coarser than $\sim$ and depends on the class Γ and the solution Φ. When we pick one propositional function $\tilde{p} \in \mathcal{P}$, an equivalence class $E_{\tilde{p}}(\Gamma,\Phi)$ of $\tilde{p}$ by $\sim_\Phi$ is called an *axiom* on Γ for Φ with respect to $\tilde{p}$. Moreover if $p((N,v),\Phi(N,v)) = 1$ for all $p \in E_{\tilde{p}}(\Gamma,\Phi)$, for all $(N,v) \in \Gamma$, we say that the solution Φ satisfies the axiom $E_{\tilde{p}}(\Gamma,\Phi)$ with respect to $\tilde{p}$.

We define the *dual* of a propositional function p, $\mathbf{D}p$, by $\mathbf{D}p((N,v),Q) = p((N,\mathbf{D}v),Q)$ for all $(N,v) \in \Gamma^A$ and $Q \subseteq \mathcal{R}^N$. We call the equivalence class $E_{\mathbf{D}p}(\Gamma,\Psi)$ in $\mathcal{P}$ the *dual axiom* on Γ for Ψ of $E_p(\Gamma,\Phi)$ with respect to p, and denote it by $\mathbf{D}E_p(\Gamma,\Psi)$.

Proposition 2 *Let p be a propositional function. Then a solution Φ on Γ satisfies an axiom $E_p(\Gamma,\Phi)$ if and only if the dual solution $\mathbf{D}\Phi$ satisfies the dual axiom $\mathbf{D}E_p(\mathbf{D}\Gamma,\mathbf{D}\Phi)$.*

AXIOM PO(Γ,Φ) [Pareto optimality]: $x(N) = v(N)$ for any $x \in \Phi(N,v)$ and (N,v) in Γ.
AXIOM NE(Γ,Φ) [non-emptiness]: $\Phi(N,v) \neq \emptyset$ for any (N,v) in Γ.
AXIOM IR(Γ,Φ) [individual rationality]: $x_i \geq v(\{i\})$ for all $i \in N$, for any $x \in \Phi(N,v)$ and $(N,v) \in \Gamma$.
AXIOM DIR(Γ,Φ): $x_i \geq v(N) - v(N \setminus \{i\})$ for all $i \in N$, for any $x \in \Phi(N,v)$ and $(N,v) \in \Gamma$.
AXIOM RGP(Γ,Φ) [reduced game property]: Let $\mathbf{R}^{x,j}$ be a reduction operator and Γ be a set of games which is closed for the operator $\mathbf{R}^{x,j}$ for any $x \in \mathcal{R}$ and $j \in \mathcal{N}$. For a game (N,v) in $\Gamma, n \geq 2$, $j \in N$, if $y \in \Phi(N,v)$, then $y|_{\mathcal{R}^{N\setminus\{j\}}} \in \Phi(N \setminus \{j\}, \mathbf{R}^{y_j,j}v)$,

Proposition 3 *Let $\Psi = \mathbf{D}\Phi$. If Φ satisfies $E_{rgp}(\Gamma,\Phi;\mathbf{R}^{x,j})$ (RGP(Γ,Φ) axiom given by a reduction operator $\mathbf{R}^{x,j}$), then Ψ satisfies $E_{rgp}(\mathbf{D}\Gamma,\Psi;\overline{\mathbf{R}^{x,j}})$ (RGP$(\mathbf{D}\Gamma,\Psi)$ axiom given by $\overline{\mathbf{R}^{x,j}}$), where $\overline{\mathbf{R}^{x,j}}v = \mathbf{D}(\mathbf{R}^{x,j}(\mathbf{D}v))$.*

3. Dual Axiomatization of the Core

AXIOM NE(Γ^C,Φ) [non-emptiness for Γ^C]: $\Phi(N,v) \neq \emptyset$ for any (N,v) in Γ^C where $\Gamma^C = \{(N,v) \in \Gamma^A | C(N,v) \neq \emptyset\}$.
AXIOM IR'(Γ^{AC},Φ): $x_i \leq v(\{i\})$ for all $i \in N$, for any $x \in \Phi(N,v)$ and $(N,v) \in \Gamma^{AC}$.

Theorem 4 (Tadenuma) *The anti-core $AC(N,v)$ on Γ^{AC} is the unique solution which satisfies axioms PO(Γ^{AC},Φ), NE(Γ^{AC},Φ), IR'(Γ^{AC},Φ) and RGP(Γ^{AC},Φ) with respect to the operator $\mathbf{R}^{x,j}v$: $(\mathbf{R}^{x,j}v)(S) = v(S \cup \{j\}) - x$ for any $S \subset N \setminus \{j\}, S \neq \emptyset$.*

The dual of the reduction operator of this RGP(Γ^{AC},Φ) is, for $S \subset N \setminus \{j\}$, $S \neq \emptyset$, $(\overline{\mathbf{R}^{x,j}}v)(S) = v(S)$. The dual of axiom IR'$(\Gamma^{AC},\Phi)$ is called UP(Γ^{AC},Φ):

AXIOM UP(Γ^{AC},Φ) [upper boundness]: $x_i \leq v(N) - v(N \setminus \{i\})$ for all $i \in N$, for any $x \in \Phi(N,v)$ and $(N,v) \in \Gamma^{AC}$.
AXIOM SGR(Γ^{AC},Φ) [sub-grand rationality]: $x(N \setminus \{i\}) \geq v(N \setminus \{i\})$ for all $i \in N$, for any $x \in \Phi(N,v)$ and $(N,v) \in \Gamma^{AC}$.

Proposition 5 *The core $C(N,v)$ on Γ^C is the unique solution which satisfies axioms PO(Γ^C,Φ), NE(Γ^C,Φ), UP(Γ^C,Φ) (or SGR(Γ^C,Φ)) and RGP(Γ^C,Φ) with respect to the operator $\mathbf{R}^{x,j}v$: $(\mathbf{R}^{x,j}v)(S) = v(S)$ for any $S \subset N \setminus \{j\}, S \neq \emptyset$.*