Uniqueness of the Equilibrium in Non-cooperative Games with a Continuum of Players

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1 Introduction

Schmeidler (1973) formulated the model of non-cooperative games with a continuum of players to describe the social situation with many players and showed that there exists pure equilibria when each player's payoff depends on an *average strategy.* Rath (1991) reformulated this case and showed the simple proof of existence of pure strategy equilibria.

Watanabe (1996) studied the sufficient condition for uniqueness of the equilibrium in Schmeidler's model. When each player has two strategies, we showed conditions for the uniqueness of the equilibrium. When each player has more than two strategies, we showed conditions for the uniqueness of the *interior equilibrium.* In this abstract and the presentation, we show the latter case.

2 Definitions

Let $(T, \mathcal{T}, \lambda)$ be a measure space of players where a nonempty set T denotes a set of players, $\mathcal{T}$ is a σ-field and λ is a finite atomless measure on $\mathcal{T}$. Each player has n strategies, and e^j, which is the j-th unit vector in R^n, denotes the j-th strategy of the players. Let $E = \{e^1, \ldots, e^n\}$ be the set of strategies. A strategy profile f is a measurable function from T to E. Let F be the set of strategy profiles and for any $f \in F$ let $s(f) = \int_T f d\lambda$. $s(f)$ is called an average strategy. Let S be the set of all average strategies. Let $\mathcal{U}$ denotes the set of real valued continuous functions defined on $E \times S$. A game g is defined as a function from T to $\mathcal{U}$. We say that $f \in F$ is an equilibrium of a game g if for almost all $t \in T$,

$$g(t)(f(t), s(f)) \geq g(t)(e^i, s(f))$$

for any $e^i \in E$.

We restrict the class of the games to normalized games. A game g is said to be a normalized game if $g(t)(e^n, q) = 0$ for any $t \in T$ and $q \in S$. Any game $\bar{g}$ can be normalized to the game g by

$$g(t)(e^j, q) = \hat{g}(t)(e^j, q) - \hat{g}(t)(e^n, q).$$

For any game, the normalized game does not change the best response structure and the equilibria from the original ones. Hence, we consider only normalized games.

3 Results

Although we consider uniqueness of the equilibrium, we consider that a strategy profile is identify to another strategy profile which has the same values as the profile outside the null sets. Formally we define uniqueness of the interior equilibrium as follows.

Definition 1 *For any game g, we say that the interior equilibrium of g is unique if for any equilibrium f and f' in g,*

$$\lambda(\{t \in T | f(t) \neq f'(t)\}) = 0.$$

whenever $s(f)_i > 0$ *and* $s(f')_i > 0$ *for any* $i \in \{1, \ldots, n\}$.

Let Γ be a correspondence from S to S defined by

$$\Gamma(q) = \{\int f d\lambda | f(t) \in B(t,q)\}$$

where

$$B(t,q) = \{e^i \in E | g(t)(e^i, q) \geq g(t)(e^j, q) \text{ for any} e^j \in E\}$$

Thus, Γ is the best response correspondence for an average strategy.

Condition N *A game* g *satisfies Condition* N *if for any* $e^i, e^j \in E$, $e^i \neq e^j$ *and any* $q \in S$,

$$\lambda(\{t \in T | g(t)(e^j, q) = g(t)(e^i, q)\}) = 0$$

Lemma 1 *If a game* g *satisfies condition* N *and the interior fixed point of* Γ *of* g *is unique, then equilibria of the game* g *is unique.*

We introduce the two notations to define the conditions which imply the unique interior fixed point of Γ. For any $\theta \geq 0$ and $k \in \{1, \ldots, n-1\}$, we define $\Delta^k(\theta)$ by $\Delta^k(\theta) = \theta(e^k - e^n)$. For any $\theta > 0$ and $i \in \{1, \ldots, n-1\}$ we define $\theta \otimes q$ by

$$\theta \otimes q = (\theta q_1, \theta q_2, \ldots, \theta q_{n-1}, 1 - \theta \sum_{j=1}^{n-1} q_j).$$

Condition R *A normalized game* g *satisfies condition* R *if for any* $t \in T$, $q \in S$, $i \in \{1, \ldots n-1\}$, $j \in \{1, \ldots n\}$, $k \in \{1, \ldots, n-1\}$ *satisfying* $i \neq k$ *and any* $\theta > 0$ *satisfying* $q + \Delta^k(\theta) \in S$, *we have* $g(t)(e^i, q + \Delta^k(\theta)) \geq g(t)(e^j, q + \Delta^k(\theta))$ *whenever* $g(t)(e^i, q) \geq g(t)(e^j, q)$.

Condition H *A normalized game* g *satisfies condition* H *if for any* $t \in T, q \in S, e^i, e^j \in E$ *and any* $\theta > 0$ *satisfying* $\theta \otimes q \in S$, *we have* $g(t)(e^i, \theta \otimes q) > g(t)(e^j, \theta \otimes q)$ *whenever* $g(t)(e^i, q) > g(t)(e^j, q)$

Lemma 2 *If a normalized game* g *satisfies condition* R *and* H, *interior fixed points of* Γ *of the game is unique.*

Theorem 1 *If a normalized game* g *satisfies condition* N, R *and* H, *then equilibria of the game* g *is unique.*

We can find the class of the functions of g which satisfy R and H.

Condition G *A normalized game* g *satisfies condition* G *if for any* $t \in T$, *there exists a positive function* $\bar{h}_t(q_1, \ldots, q_{n-1})$ *and a non-increasing homogeneous function with* m *degree* $h^i_t(q_i)$ $(i = 1, \ldots, n-1)$ *such that*

$$g(t)(e^i, q) = \bar{h}_t(q_1, \ldots, q_{n-1}) h^i_t(q_i).$$

Lemma 3 *If a normalized game* g *satisfies condition* G, *then the game* g *satisfies condition* R *and* H.

Theorem 2 *If a normalized game* g *satisfies condition* N *and* G, *then equilibria of the game* g *is unique.*

References

D. Schmeidler (1973) Equilibrium Points of nonatomic games. *Journal of Statistical Physics*, 7, 295–300.

K. P. Rath (1995) Representation of finite action games, *International Journal of Game Theory*, 24, 23–35.

T. Watanabe (1996), Doctoral Dissertation (in Japanese).