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Drawing a tree on parallel lines

01504250 東京理科大学 平林 隆一 Ryuichi HIRABAYASHI
01012660 東京理科大学 池辺 淑子 Yoshiko T. IKEBE *
城西大学 岩村 覚三 Kakuzo IWAMURA

1 Introduction

Suppose we are given a tree T on n vertices, and a (infinite) set of parallel lines $\{\ell_1, \ell_2, \ldots\}$ in the plane. We wish to draw the tree in the plane so that

(i): each vertex is placed on one of the parallel lines; we call the number of the line which vertex v is placed upon the *level* of v,

(ii): the line segments corresponding to edges of T do not intersect each other (except possibly at their endpoints), and

(iii): the level of vertices is nondecreasing along any path beginning from a vertex having the lowest level.

We will call such a drawing *feasible*. The object is to use the fewest number of lines possible.

This problem was first posed in [1] by Fukuhara. In this talk, we present a simple recursive algorithm to solve this problem. Moreover, the algorithm runs in time linear on the number of vertices of the tree.

2 The algorithm

Let T be a tree on n vertices. For any vertex v of T, the *degree* of v is the number of edges incident to v, and is denoted by $\delta_T(v)$. Vertices with degree one are called *leaves*, all other vertices are called *internal*. A tree with only one internal vertex is called a *star*. We wish to find a feasible drawing of T in which the number of lines on which vertices are placed is minimum. We will call such a feasible drawing *optimal*. Obviously, there may be many optimal drawings of the same tree. For example, if we have an optimal drawing of T in which the level of a leaf v is not maximum, then the drawing obtained by 'raising' the level of v to the maximum is also optimal, but in a sense, unnatural. In order to eliminate such 'unnatural' drawings, we introduce the concept of standard optimal drawings. More explicitly, we say that an optimal drawing is *standard* if the level of any vertex is minimum, when the levels of all other vertices are fixed. In the sequel, we observe some properties of standard optimal drawings, and use these properties to construct a standard optimal drawing for any given tree. Before going into details, we first note that

(a): any vertex with degree two which is adjacent to either a leaf, or another vertex of degree two may be contracted, and

(b): if T is a star or path, then T may be trivially drawn.

Here by contracting a vertex v with degree two, we mean we replace the vertex v and the two edges (u, v) and (v, w) incident to v, by the edge (u, w). Henceforth, we will assume that any internal vertex of T adjacent to a leaf has degree at least three, there are no consecutive vertices of degree two, and that T contains a path of length three or more. We also state the following, rather obvious facts:

Fact 1. *The vertices of level one form a path.*

Fact 2. *In a standard optimal drawing of T, there are at least two vertices of level one.*

We now subdivide the internal vertices of T. For any internal vertex v of T, we say that v is a *semi-leaf* if the number of leaves adjacent to v is equal to $\delta_T(v) - 1$. The following lemmas pertaining to semi-leaves hold.

Lemma 2.1. *Let v be a semi-leaf of T with $\delta_T(v) \geq 4$. Then, in any standard optimal drawing of T, the levels of all leaves adjacent to v are either equal to, or one more than that of v, and there exists a leaf whose level is exactly one more than that of v.*

Lemma 2.2. *Let v be a semi-leaf of T with $\delta_T(v) = 3$. Then, in any standard optimal drawing of T, the levels of the two leaves adjacent to v are either both equal to the level of v, or the level of one leaf is equal to the level of v, and the level of the other is equal to the level of v plus one*

In light of these lemmas, we will call a semi-leaf with degree at least four an *h2-leaf* (for height-two leaf), and a semi-leaf with degree three, a *quasi-h2-leaf.* These lemmas say that we may find an optimal drawing of T from the tree T' obtained by shrinking all leaves adjacent to a semi-leaf of T into that semi-leaf Here, by shrinking a leaf into the internal vertex v it is adjacent to, we mean that we delete the leaf and its unique incident edge, and make v 'remember' this deletion (perhaps by maintaining a list containing this leaf). We now observe some properties of semi-leaves in T'. Note that for any semi-leaf v of T', at least one of the leaves adjacent to v must be an h2-leaf or a quasi-h2-leaf.

Lemma 2.3. *Let v be a semi-leaf of T' of which none of the adjacent leaves is an h2-leaf, and let U be the set of vertices consisting of the leaves of v, and any leaves of T which have been shrunk into them. Then, for any standard optimal drawing of T, the level of any vertex of U is either equal to the level of v, or the level of v plus one, and there is at least one vertex which attains the latter value.*

Lemma 2.3 says that any semi-leaf of T' which has no h2-leaves adjacent to it may be shrunk along with its adjacent leaves into an h2-leaf. Let T'' be the tree obtained from T' by performing all such shrinkings. It is easily seen that any semi-leaf of T'' always has an h2-leaf adjacent to it.

Lemma 2.4. *Let U_1 be the set of leaves of T'' which are not h2-leaves, together with any vertices which have been shrunk into them, and likewise let U_2 be the set of h2-leaves, and any shrunken vertices. Then, the level of any vertex of U_1 is less than or equal to the maximum level of the vertices in U_2.*

Roughly speaking, this lemma says that we may shrink all leaves of T'' which are not h2-leaves into the internal vertices they are adjacent to. If we do this, this will result in a tree whose leaves are all h2-leaves, and which may have many internal vertices of degree two. Moreover, an optimal drawing of the original tree T can easily be constructed from an optimal drawing of this resulting tree, hence it suffices to find such an optimal drawing. To do this, we contract such vertices of degree two as allowed, to obtain the tree T_2. and simply perform the same operations recursively to obtain T_3. except that, instead of h2-leaves and quasi-h2-leaves, we have h3-leaves and quasi-h3-leaves. And so in general we proceed, creating $T_4, T_5, \ldots,$ calling semi-leaves of T_k either h$(k+1)$-leaves or quasi-h$(k+1)$-leaves, until we finally arrive at a path or star.

To obtain a standard optimal drawing from the final T_k (which is either a path or a star), we draw T_k in the obvious way, and then retrace the above procedure in reverse, expanding and drawing in the appropriate manner.

Finally we evaluate the time complexity of this method.

Lemma 2.5. *T_{k+1} can be obtained from T_k in $O(|T_k|)$ time, where $|T_k|$ is the number of vertices of T_k. Also, a standard optimal drawing of T_k can be obtained from one of T_{k+1} in $O(|T_k|)$ time,*

Lemma 2.6. $|T_{k+1}| \leq \frac{2}{3}|T_k|$.

Combining these, we obtain

Theorem 2.7. *The proposed algorithm correctly finds a standard optimal drawing in $O(n)$ time.*

References

[1] Fukuhara, T. (1990), "Structural study of a graph drawing problem on k parallel lines," Master Thesis (in Japanese), Dept. of Engineering and Electronics, Tokyo Institute of Technology, Tokyo.