1999年度日本オペレーションズ・リサーチ学会
春季研究発表会

An Optimal Stopping Problem for a Geometric Brownian Motion with Poissonian Jumps

Masamitsu OHNISHI (Graduate School of Economics, Osaka University)

1 Introduction

This paper examines an optimal stopping problem for a geometric Brownian motion with random jumps. It is assumed that jumps occur according to a time-homogeneous Poisson process and the relative amplitudes of these sizes are independent and identically distributed. The objective is to find an optimal stopping time of maximizing the expected discounted terminal reward which is defined as a power function of the stopped state. By applying what is called smooth pasting technique, we derive almost explicitly an optimal stopping rule of a threshold type and the optimal value function of the initial state. That is, we express the critical state of the optimal stopping region and the optimal value function by formulae which include only given problem parameters except an unknown to be uniquely determined by a solution of a nonlinear equation.

2 Description of Problem

Let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space, and consider the following random elements which are defined on this space:

$\mathcal{W} = (W_t;\ t \in \mathcal{R}_+)$: a standard Brownian motion.

$\mathcal{N} = (N_t;\ t \in \mathcal{R}_+)$: a time-homogeneous Poisson counting process with intensity $\lambda \geq 0$.

$\mathcal{U} = (U_i;\ i \in \mathcal{Z}_{++})$: a sequence of independent and identically distributed $(-1, +\infty)$-valued random variables. Their generic random variable is denoted by U and their common cumulative distribution function is denoted by F_U. It is assumed that it has a finite mean m_U.

Furthermore, we assume these random elements are mutually independent. Now, we let

$\mathcal{T} = (T_i;\ i \in \mathcal{Z}_+)$: the sequence of the event times of the Poisson counting process $\mathcal{N}$.

and consider a right-continuous $\mathcal{R}_{++}$-valued stochastic process $\mathcal{X} = (X_t;\ t \in \mathcal{R}_+)$ described as follows.

(D1) On the time interval $[T_i, T_{i+1})$ $(i \in \mathcal{Z}_+)$, for some constants μ and $\sigma \geq 0$, it follows the following stochastic differential equation:

$$dX_t = X_t\left(\mu dt + \sigma dW_t\right). \tag{2.1}$$

(D2) At every event time T_i $(i \in \mathcal{Z}_{++})$ of the Poisson counting process $\mathcal{N}$, $\mathcal{X}$ jumps in a random size whose proportion is given by U_i, that is,

$$X_{T_i} = X_{T_i-}(1 + U_i). \tag{2.2}$$

Then, since the state X_t at time instant $t \in [T_i, T_{i+1})$ $(i \in \mathcal{Z}_+)$ is represented by

$$X_t = X_{T_i} \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \tag{2.3}$$

we can show, by induction in $i \in \mathcal{Z}_+$, for any time instant $t \in \mathcal{R}_+$,

$$X_t = X_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}\left[\prod_{i=1}^{N_t}(1 + U_i)\right] \tag{2.4}$$

(see, e.g., Lamberton and Lapeyre [3]). In the sequel, we denote the X_t of eq. (2.4) when the initial state is $X_0 = x$ $(\in \mathcal{R}_{++})$ by X_t^x for notational convenience.

For this state process $\mathcal{X}$, let $p > 0$, $q > 0$, and $\beta \geq 0$ be constants, and define the terminal reward function by

$$R(x) := px^\beta - q, \quad x \in \mathcal{R}_{++}. \tag{2.5}$$

Now, let us consider the optimal stopping problem whose objective is to find an optimal stopping time τ of attaining the following supremum of the expected discounted terminal reward:

$$v^*(x) := \sup_\tau E\left[e^{-\alpha\tau} R(X_\tau^x)\, 1_{\{\tau < +\infty\}}\right], \quad x \in \mathcal{R}_{++}, \tag{2.6}$$

where $\alpha > 0$ is a discount rate, and the sup of the right hand side of eq. (2.6) is taken over the set of all stopping times with respect to the state process $\mathcal{X}$.

3 Analysis

We first introduce the infinitesimal generater L of the Markovian state process $\mathcal{X}$ as follows: for twice continuously differentiable function $w : \mathcal{R}_{++} \to \mathcal{R}$,

$$[Lw](x) := \lim_{h \downarrow 0+} \frac{e^{-\alpha h} E\left[w(X_h^x)\right] - w(x)}{h}, \quad x \in \mathcal{R}_{++}. \tag{3.1}$$

Then, by Itô formula and properties of Poisson process, we have

$$[Lw](x) \quad = \quad \frac{1}{2}\sigma^2 x^2 w''(x) + \mu x w'(x) - \alpha w(x) \tag{3.2}$$

$$+\lambda\left(\int_{-1}^{+\infty} w((1+u)x)dF_U(u) - w(x)\right).$$

Now, let us consider a functional equation

$$[Lw](x) = 0, \quad x \in \mathcal{R}_{++}, \tag{3.3}$$

where $w : \mathcal{R}_{++} \to \mathcal{R}$ is an unknown function to be determined. In order to solve this functional equation, for two real numbers a and b, we apply a trial solution function of the form

$$w(x) = ax^b, \quad x \in \mathcal{R}_{++}. \tag{3.4}$$

Substituting it into eq. (3.3), we have

$$[Lw](x) = w(x)g(b) = 0, \quad x \in \mathcal{R}_{++}, \tag{3.5}$$

where the function $g : \mathcal{R} \to \mathcal{R}$ is defined by

$$\begin{aligned} g(b) \ := \ & \frac{1}{2}\sigma^2 b^2 + \left(\mu - \frac{1}{2}\sigma^2\right) b - \alpha \qquad (3.6) \\ & + \lambda\left(\int_{-1}^{+\infty} (1+u)^b dF_U(u) - 1\right), \quad b \in \mathcal{R}. \end{aligned}$$

Assumption (A1) $g(1) = \mu - \alpha + \lambda m_U \leq 0.$ □

Lemma 3. 1 Let us assume (A1). Then, the nonlinear equation $g(b) = 0$ has two distinct real roots, the larger one, b_+ of which satisfies

$$b_+ \geq 1. \tag{3.7}$$

□

Assumption (A2) $0 < \beta < b_+.$ □

Now, let us define a function $w^* : \mathcal{R}_{++} \to \mathcal{R}$ by

$$w^*(x) := \begin{cases} w(x) = a^* x^{b_+}, & 0 < x < x^*, \\ R(x) = px^\beta - q, & x^* \leq x, \end{cases} \tag{3.8}$$

where $a^* > 0$ and $x^* > 0$ are constants which are uniquely determined by the following simultaneous equations (see Dixit [1], Dixit and Pindyck [2]):

Value Matching Condition:

$$w(x^*) = R(x^*); \tag{3.9}$$

Smooth Pasting Condition:

$$w'(x^*) = R'(x^*). \tag{3.10}$$

That is,

$$a^* = q\left(\frac{q}{p}\right)^{-\frac{b_+}{\beta}} \frac{\beta}{b_+ - \beta}\left(\frac{b_+}{b_+ - \beta}\right)^{-\frac{b_+}{\beta}} \tag{3.11}$$

$$x^* = \left(\frac{q}{p}\right)^{\frac{1}{\beta}}\left(\frac{b_+}{b_+ - \beta}\right)^{\frac{1}{\beta}}. \tag{3.12}$$

Assumption (A3) $F_U(0) = 1.$ □

Assumption (A4) $\frac{1}{2}\sigma^2\beta + \left(\mu - \frac{1}{2}\sigma^2 - \frac{\alpha}{b_+}\right) \leq 0.$ □

Lemma 3. 2 Let us assume (A1), (A2), (A3), and (A4). Then, the function $w^* : \mathcal{R}_{++} \to \mathcal{R}$ satisfies the following properties (P1), (P2), (P3), (P4), and (P5):

(P1) For any $x \in \mathcal{R}_{++}$ and $t \in \mathcal{R}_+$,

$$E[|w^*(X_t^x)|] < +\infty; \tag{3.13}$$

$$E\left[\int_0^t e^{-\alpha s}|[Lw^*](X_s^x)|\,ds\right] < +\infty. \tag{3.14}$$

(P2) For any $x \in \mathcal{R}_{++}$,

$$w^*(x) \geq R(x). \tag{3.15}$$

(P3) $w^*(x)$ is strictly increasing in x.

(P4) For any $x \in \mathcal{R}_{++}$ $(x \neq x^*)$,

$$[Lw^*](x) \leq 0. \tag{3.16}$$

(P5) For any $x \in \mathcal{R}_{++}$, either of ineqs. (3.15) or (3.16) holds with equality. □

Theorem 3. 1 Let us assume (A1), (A2), (A3), and (A4). The function $w^* : \mathcal{R}_{++} \to \mathcal{R}$ is the optimal value function, that is,

$$v^*(x) = w^*(x), \quad x \in \mathcal{R}_{++}. \tag{3.17}$$

Moreover, the optimal stopping region S^* $(\subset \mathcal{R}_{++})$ and the optimal stopping time τ^* are given by the followings:

$$S^* := \{x \in \mathcal{R}_{++} : w^*(x) = R(x)\} = [x^*, +\infty) \tag{3.18}$$

$$\tau^* := \inf\{t \in \mathcal{R}_+ : X_t^x \in S^*\}. \tag{3.19}$$

□

References

[1] Dixit, A., *The Art of Smooth Pasting*, Harwood Academic Publishers, Switzerland, 1993.

[2] Dixit, A. and Pindyck, R., *Investment under Uncertainty*, Princeton University Press, New Jersey, 1994.

[3] Karatzas, I. and Shreve, S. E., *Brownian Motion and Stochastic Calculus*, Springer–Verlag, 1988.

[4] Lamberton, D. and Lapeyre, B. (Translated by Rabeau, N. and Mantion, F.), *Introduction to Stochastic Calculus Applied to Finance*, Chapman & Hall, London, 1996.

[5] Tsujimura, M., *Price Analysis of Tradable Emission Permits of CO_2 by a Real Option Model*, Master Thesis, Graduate School of Economics, Osaka University, 1998.