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Approximation algorithm for maximum independent set problems on unit disk graphs

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1. Preliminaries

Unit disk graphs are the intersection graphs of equal sized circles in the plane. Given a point-set $P \subseteq \mathrm{R}^2$ the unit disk graph defined by P, denoted by $G(P)$, is an undirected graph (P, E) with vertex set P and edge set E satisfying that $E = \{\{p_i, p_j\} \mid p_i, p_j \in P,\ \|p_i - p_j\| \leq 1\}$. The unit disk graphs provide a graph-theoretic model for broadcast network and for some problems in computational geometry.

In this paper, we consider the maximum independent set problem on the unit disk graph. The maximum independent set problem defined on unit disk graph is $\mathcal{NP}$-hard [1] and there exists a (1/3)-approximation algorithm [3]. We propose an approximation algorithm for the independent set problem on a unit disk graph which finds a $(1 - 1/r)$-approximation solution in $\mathrm{O}(|P|^{4\lceil 2(r-1)/\sqrt{3}\rceil})$ time. It is easy to extend our algorithm for solving weighted independent set problem on unit disk graph.

2. Slab with a Fixed Width

In this section, we assume that the point-set P is contained in the region $S_k = \{(x, y) \in \mathrm{R}^2 \mid 0 \leq y < k\}$.

Here we assume that the width k of the slab is a fixed constant.

For any point subset $P' \subseteq P$, we denote the value $\min\{\lfloor x \rfloor \mid (x, y) \in P'\}$ by $\min P'$. A subset of points $B \subseteq P$ is called an *independent block* when B is an independent set of the unit disk graph $G(P)$ and each point (x, y) in P' satisfies $\lfloor x \rfloor = \min B$. Let $\mathcal{B}(P)$ be the family of all the independent blocks of P.

Now we introduce an auxiliary graph which is helpful for finding a maximum independent set of $G(P)$. The auxiliary graph, denoted by $A(P)$, is a directed graph with node set $\{s, t\} \cup \mathcal{B}(P)$ and arc (directed edge) set

$$\begin{aligned}&\{(s, B) \mid \forall B \in \mathcal{B}(P)\} \cup \{(B, t) \mid \forall B \in \mathcal{B}(P)\}\\ &\cup \{(B, B') \in \mathcal{B}(P) \times \mathcal{B}(P) \mid\\ &\quad (\min B) < (\min B') \text{ and}\\ &\quad B \cup B' \text{ is an independent set}\}.\end{aligned}$$

Then it is clear that for any directed s-t path in the auxiliary graph the union of independent blocks corresponding to internal nodes is an independent set of $G(P)$. Conversely, for any independent set in $G(P)$ there exists a corresponding directed s-t path in $A(P)$. For each non-terminal node (independent block) of the auxiliary graph, we associate the weight which is equal to the size of the corresponding independent block. Thus, the maximum independent set problem on$G(P)$ is reduced to the problem for finding the longest directed path in the auxiliary graph.

We can generate all the indepent blocks by applying an enumeration algorithm for maximal independent sets in [4] which requires $\mathrm{O}(|P|^3|\mathcal{B}(P)|)$ time. The ordinary dynamic programming method finds the longest path in linear time with respect to the number of arcs [2]. From the above, the total time complexity of our algorithm is bounded by $O(|P|^3|\mathcal{B}(P)| + |\mathcal{B}(P)|^2)$. When we denote the size of maximum independent block by α, $|\mathcal{B}(P)| = O(|P|^\alpha)$. If we consder the non-trivial problem instances satisfying that $\alpha \geq 2$, the total time complexity of our algorithm is bounded by $O(|P|^{2\alpha})$

The following lemma shows a simple upper bound of the value α.

Lemma We define the value α_k by

$$\alpha_k = \max\{|P'| \mid P' \subseteq [0,1) \times [0,k), \\ \forall \boldsymbol{p}_i, \forall \boldsymbol{p}_j \in P', \boldsymbol{p}_i \neq \boldsymbol{p}_j \Rightarrow ||\boldsymbol{p}_i - \boldsymbol{p}_j|| > 1\}$$

Then $\alpha_k \leq 2\left\lceil 2k/\sqrt{3} \right\rceil$.

The time complexity of our algorithm is bounded by $O(|P|^{4\lceil 2k/\sqrt{3}\rceil})$ when we apply our algorithm to the problem defined on the slab whose width is equal to k.

3 Approximation algorithm

For any $s \in \{0,1,\ldots,r-1\}$, the region $\{(x,y) \in \mathrm{R}^2 \mid s \leq (y \bmod r) < s+1\}$ is denoted by T_s. We construct point-subsets $P_0, P_1, \ldots, P_{r-1}$ defined by $P_s = P \setminus T_s$. Next we solve the maximum independent set problems defined on the graphs $G(P_0), G(P_1), \ldots, G(P_{r-1})$ and output one of the best solutions. The size of the output independent set is greater than or equal to $(1-1/r)z^*$. It is because, a maximum independent set P^* satisfies that for any $s \in \{0,1,\ldots,r-1\}$, $P^* \setminus T_s \subseteq P \setminus T_s$ and $\max\{|P^* \setminus T_s| \mid s \in \{0,1,\ldots,r-1\}\} \geq (1-1/r)|P^*|$.

By using the simple upper bound in the previous lemma, the time complexity of our algorithm is bounded by $O(r|P|^{2\alpha_{r-1}}) = O(r|P|^{4\lceil 2(r-1)/\sqrt{3}\rceil})$.

It is easy to extend our algorithm for the weighted version.

Lastly, we consider the memory space. For any integer k', we denote the set of independent blocks $\{B \in \mathcal{B}(P) \mid k' = \min B\}$ by $\mathcal{B}_k(P)$. If we apply the ordinary labeling procedure for solving the longest path problem (see [2]), we only need to maintain consecutive three families of independent blocks $\mathcal{B}_{k'-1}(P) \cup \mathcal{B}_{k'}(P) \cup \mathcal{B}_{k'+1}(P)$ for labeling independent blocks in $\mathcal{B}_{k'+1}(P)$. When we label an independent block B in $\mathcal{B}_{k'+1}(P)$, we generate arcs in $A(P)$ entering to B one by one. Thus the required memory space is bounded by $O(\mathcal{B}(P))$.

References

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