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A Generalized Optimum Requirement Spanning Tree Problem with a Monge-like Property

01013281　札幌大学　穴沢　務　ANAZAWA Tsutomu

1 Introduction

We begin by introducing the *optimum requirement spanning tree problem* (ORST problem) studied by Hu [5], which has motivated our studies. Let $V = \{0, 1, \ldots, n-1\}$ be a set of n vertices, $\binom{V}{2}$ the set of all pairs of distinct vertices in V, and $\mathcal{T}$ the whole set of undirected spanning trees on V. A tree $T \in \mathcal{T}$ with an edge set $E = \{(v,u) | v, u \in V\}$ is denoted by $T = (V, E)$ conventionally. Assume that a nonnegative value r_{vu} is given to each $\{v, u\} \in \binom{V}{2}$, where $r_{vu} = r_{uv}$ holds. Hu [5] defined an ORST to be a tree $T \in \mathcal{T}$ which minimizes $f(T) = \sum_{\{v,u\}\in\binom{V}{2}} d(v,u;T) r_{vu}$, where $d(v,u;T)$ is the length of the path between v and u on T. He showed that a tree minimizing f is obtained by the Gomory-Hu algorithm [4] when degrees of vertices are not restricted. ORST problem has been extened by Anazawa et al. [2, 3] and Anazawa [1] in different manners. The aim of this paper is to generalize the problem and results discussed in the literature. For generalization, we define a *dummy vertex* as a vertex whose number is greater than $n-1$, and assme that $r_{vu} = 0$ holds if v or u is dummy. Here we consider the problem to find a tree $T \in \mathcal{T}$ minimizes a function $f_g(T) = \sum_{\{v,u\}\in\binom{V}{2}} g(d(v,u;T)) r_{vu}$ under constraint that $\deg(v) \le l_v$ holds for all $v \in V$ (called *maximum degree constraint*), where $g(x)$ be an arbitrary real-valued function of real variable x such that it is strictly increasing on $[0, n-1)$, and l_v is given to each vertex $v \in V$. We call this problem a *generalized optimum requirement spanning tree problem* (GORST problem), and a solution to this problem an *f_g-optimum tree.* Our main assertion on GORST problem in this paper is the following:

Main Theorem *Suppose that $l_0 \ge l_1 \ge \cdots \ge l_{n-1}$ and $\sum_{v=0}^{n-1} l_v \ge 2(n-1)$ hold. If $\{r_{vu}\}$ satisfies $r_{vu} + r_{v'u'} \ge r_{vu'} + r_{v'u}$ for all 4-tuple $\{v, v', u, u'\}$ $(v < v', u < u')$ such that r_{vu}, $r_{v'u'}$, $r_{vu'}$ and $r_{v'u}$ are all defined (called* Monge-like property*), then T^* defined below is f_g-optimum.*

Definition of T^*: We set $s_{-1} = 0$, $s_0 = l_0$, $s_u = s_{u-1} + (l_u - 1)$ $(u = 1, 2, \ldots)$ and let N be the minimum integer satisfying $n - 1 \le s_{N-1}$; also we define a function π on a set $\{1, 2, \ldots, n-1\}$ by

$$\pi(v) = \begin{cases} u & \text{if } s_{u-1} + 1 \le v \le s_u \quad \text{for } u = 0, 1, 2, \ldots, N-2 \\ N-1 & \text{if } s_{N-2} + 1 \le v \le n-1 \end{cases},$$

and let $E^* = \{e_1, e_2, \ldots, e_{n-1}\}$, where $e_v = (\pi(v), v)$ $(v = 1, 2, \ldots, n-1)$. Then we obtain $T^* = (V, E^*)$.

As the name indecated, Monge-like property is closely related to the Monge property, which is originally discussed in the classical Hitchcock transportation problem, and is known to make some NP-hard problems efficiently solvable (see [6]).

2 Preliminaries and lemmas

For a graph $G = (V, E)$ and a subset $U \subset V$, a subgraph $G \cap U$ is defined by a forest $G' = (U, E')$, where $E' = \{(v,u) \in E | v, u \in U\}$; while a subgraph $G \setminus U$ is defined by a forest $G'' = (V \setminus U, E'')$, where $E'' = \{(v,u) \in E | v, u \in V \setminus U\}$. For a rooted tree $T = (V, E) \in \mathcal{T}$ and a vertex $v \in V$, let $\chi(v) = \{u | v \text{ is the parent of } u\}$. For a path $P = (u_1, u_2, \ldots, u_k)$ of a tree $T = (V, E) \in \mathcal{T}$, let F be a forest defined by $F = (V, E \setminus \{(u_1, u_2), (u_2, u_3), \ldots, (u_{k-1}, u_k)\})$, and $T(u_i) = (V(u_i), E(u_i))$ $(i = 1, \ldots, k)$ the connected components each of which contains u_i. We regard u_i as the root of $T(u_i)$ in the sequel. An edge (v, u) such that v or u is a dummy vertex is called a *dummy edge.* For a tree $T = (V, E)$, we will sometimes construct another tree $\tilde{T} = (\tilde{V}, \tilde{E})$ satisfying $\tilde{T} \setminus \{\text{dummy vertices}\} = T$. Then it is obvious that $f_g(\tilde{T}) = f_g(T)$ holds. Suppose that a tree $T = (V, E) \in \mathcal{T}$ satisfies the maximum degree constraint and a path $P = (u_1, \ldots, u_k)$ $(k = 2$ or $3)$ of T is given. Then we construct $\tilde{T} = (\tilde{V}, \tilde{E})$ by adding dummy vertices and edges to T, and simultaneously introduce an isomorphism σ_P as follows: Let us construct $\tilde{T}(u_i) = (\tilde{V}(u_i), \tilde{E}(u_i))$ $(i = 1, k)$ by adding dummy vertices and edges to $T(u_i) = (V(u_i), E(u_i))$ $(i = 1, k)$ defined for P so that we can define an isomorphism $\sigma_P : \tilde{V}(u_1) \to \tilde{V}(u_k)$ satisfying (i) $\sigma_P(u_1) = u_k$, (ii) $(\sigma_P(w), \sigma_P(w')) \in \tilde{E}(u_k)$ if and only if $(w, w') \in \tilde{E}(u_1)$, and (iii) for any $v \in \tilde{V}(u_1)$, $|\chi(v)| = |N|$ or $|\chi(\sigma_P(v))| = |N'|$ holds where $N = \{w \in \chi(v) | w \leq n-1\}$ and $N' = \{w' \in \chi(\sigma_P(v)) | w' \leq n-1\}$. We call such an isomorphism σ_P a *forced isomorphism* for P. Also for $\tilde{T}$ and σ_P defined above, we consider the following transformation of $\tilde{T}$ which may reduce the f_g value: Let $V_C = \{v \in \tilde{V}(u_1) | v > \sigma_P(v)\}$, and exchange v and $\sigma_P(v)$ for all $v \in V_C$. We call such a transformation *biasing* with respect to σ_P. Further let $\tilde{T}'$ be a tree obtained from $\tilde{T}$ by biasing and $T' = \tilde{T}' \setminus \{\text{dummy vertices}\}$.

Lemma 1 *T' is also a tree belonging to $\mathcal{T}$ and satisfies the maximum degree constraint.*

Lemma 2 *If $\{r_{vu}\}$ satisfies Monge-like property, then $f_g(T') \leq f_g(T)$ holds.*

Here we show some properties of the tree $T^* = (V, E^*)$ satisfying $l_0 \geq l_1 \geq \cdots \geq l_{n-1}$. Let $T^*_\nu = T^* \cap \{0, 1, \ldots, \nu - 1\}$ $(\nu = 1, 2, \ldots, n)$ be subtrees of the tree T^*.

Lemma 3 *For each T^*_ν $(\nu \geq 2)$, let $P = (u_1, u_2, \ldots, u_k)$ be an arbitrary path of T^*_ν satisfying $u_1 < u_k$, and let $m = \lfloor \frac{k}{2} \rfloor$ where $\lfloor x \rfloor$ is the maximum integer not exceeding x. Then $u_i < u_{k-i+1}$ and $\deg(u_i) \geq \deg(u_{k-i+1})$ hold for $i = 1, 2, \ldots, m$.*

Lemma 4 *Let T be a tree containing a subtree T^*_ν (that is, $T \cap \{0, 1, \ldots, \nu-1\} = T^*_\nu$), and $P = (u_1, \ldots, u_k)$ $(k = 2$ or $3)$ an arbitrary path of T. For the tree T and the path P, let $\tilde{T}$ be a dummies-added tree on which a forced isomorphism σ_P is defined, $\tilde{T}'$ a tree obtained from $\tilde{T}$ by biasing with respect to σ_P, and $T' = \tilde{T}' \setminus \{\text{dummy vertices}\}$. Then T' also contains T^*_ν.*

3 Proof of Main Theorem (Outline)

Let $T^* \in \mathcal{T}$ be the tree stated in Main Theorem, and $\mathcal{T}_{\text{opt}}$ the set of f_g-optimum trees. By using Lemmas 2 and 4, we can show that T^* must belong to $\mathcal{T}_{\text{opt}}$.

4 Application

Let vertices be regarded as network hosts, edges as network cables, and $\{r_{vu}\}$ as relative frequencies of communication. Also let $p(T;k)$ denote the probability that a request of communication is not realized on a tree network $T = (V, E)$ with k failures. Under some conditions, $p(T;k)$ is expressed by $\sum_{\{v,u\} \in \binom{V}{2}} g(d(v, u; T)) r_{vu}$ where

$$g(x) = 1 - \sum_{i+j=k} \frac{\binom{n-1-x}{i}\binom{n-1-x}{j}}{\binom{n}{i}\binom{n-1}{j}} \alpha_{ij}$$

and $\alpha_{ij} = \Pr\{i \text{ vertices and } j \text{ edges are broken down} | i+j \text{ failures have occured}\}$. Noting that $g(x)$ is strictly increasing on $[0, n-1)$,. We find that T^* defined above minimizes $p(T;k)$ for any k $(0 < k \leq 2n-1)$ if $\{r_{vu}\}$ satisfies Monge-like property.

References

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