Numerical Valuation of a Switched Knockout Option

02502201 北海道大学 *花田 邦生 HANADA KUNIO
01105381 北海道大学 木村 俊一 KIMURA TOSHIKAZU

1 Introduction

For a vanilla European option, the payoff at exercise can be determined by the spot price of the underlying asset, independently on its past history in the trading interval. The so-called *exotic* or *path-dependent* options have values that depend on the history of the asset price in some non-trivial way. Among various exotic options, we focus on a knockout option with an incomplete boundary in this paper.

Knockout options are contingent claims whose right to exercise is nullified when the undelying asset value crosses a certain value. The set of those values over the trading interval is called a knockout boundary. Knockout options are classified as either *up-and-out* or *down-and-out* options by the relative position between initial values of the asset price and the knockout boundary. Of course, they are classified into two basic types, *i.e.*, *call* or *put*. Hence, there are totally four different types in knockout options.

All of previous results of knockout options (*e.g.*, [3, 4]) are based on a common assumption that the knockout boundary exists in the whole trading interval from initial time to maturity. In this paper, however, we consider an incomplete knockout boundary that exists only in a certain part of the trading interval. In other words, there is a toggled *switch* in the knockout boundary; this option is equivalent to a vanilla or an ordinary knockout option according as the switch is *off* or *on*. Hence, we call it a *switched knockout option* in this paper. Obviously, the vanilla and ordinary knockout options are special cases of our switched knockout option.

Due to the page restriction, we are only concerned with the analysis of the up-and-out call type and we omit all figures in Section 3; see Hanada and Kimura [2] for details.

2 Mathematical Formulation

We use the same assumptions as those in the Black-Scholes model [1] except for knockout boundaries: Assume that the capital market is well-defined and follows the efficient market hypothesis. Let $S(t)$ denote the underlying asset price at time t and let T (≥ 0) be the maturity. Then, the process $\{S(t);\ 0 \leq t \leq T\}$ satisfies the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \quad 0 \leq t \leq T, \quad (1)$$

where μ (σ) is the drift (volatility) of the process $S(\cdot)$ and r is the risk-free interest rate, all of which are assumed to be positive constants. In (1), $\{W(t);\ 0 \leq t \leq T\}$ is the standard Brownian motion process. Also, assume that the option price written on $S(t)$, say V, is a function of $S(t)$ and t, *i.e.*, $V \equiv V(S(t),t)$ for $S(t) > 0$ and $0 \leq t \leq T$. From these assumptions and Itô's lemma, we have the partial differential equation [1]

$$\frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + rS(t)\frac{\partial V(S(t),t)}{\partial S(t)} - rV(S(t),t) + \frac{\partial V(S(t),t)}{\partial t} = 0. \quad (2)$$

For a vanilla call option with the exercise price K (> 0), the option price V satisfies the terminate condition

$$V(S(T),T) = \max(S(T) - K,\ 0), \quad (3)$$

together with the boundary conditions

$$\lim_{\xi\to\infty} \frac{V(\xi,t)}{\xi - Ke^{-r(T-t)}} = 1, \quad (4)$$

$$\lim_{\xi\to 0} V(\xi,t) = 0. \quad (5)$$

For a switched knockout option, however, these boundary conditions should be modified as follows: Let $\mathcal{I}_{\text{on}}$ be the set of time intervals where the nullified switch is on, and let $\mathcal{I}_{\text{off}} \equiv [0,\ T]\backslash\mathcal{I}_{\text{on}}$. Let $B(t)$ be the value of knockout boundary at time t and assume that $B(t) > 0$ for $t \in [0,\ T]$. Then, for the up-and-out call type, the boundary conditions should be

$$\lim_{\xi\to\infty} \frac{V(\xi,t)}{\xi - Ke^{-r(T-t)}} = 1, \quad t \in \mathcal{I}_{\text{off}},$$

$$V(\xi,t) = 0, \quad (\xi,t) \in [B(t),\infty) \times \mathcal{I}_{\text{on}}, (6)$$

$$\lim_{\xi\to 0} V(\xi,t) = 0, \quad 0 \le t \le T.$$

3 General Properties

Figures 1 and 2 illustrate the curves of the up-and-out call price $V(S(0),0)$ as a function of $S(0)$ for several knockout boundaries, where the intervals $\mathcal{I}_{\text{on}} = \emptyset$ (*i.e.* empty set) and $\mathcal{I}_{\text{on}} = [0,\ 1]$ are added for comparisons, which represent the vanilla and ordinary knockout options, respectively. Clearly, the prices of these extreme cases give upper and lower bounds for V of the switched knockout options. In Figure 1, the knockout boundaries exist in latter parts of the trading interval, whereas in Figure 2 they exist in former parts. From these figures, we see that there are significant differences between these two cases: The option prices for the former-part cases are higher and more sensitive to the length of $\mathcal{I}_{\text{on}}$ than those for the latter-part cases. No doubt, this result is due to the assumption that the process $S(\cdot)$ follows a geometric Brownian motion with continuous sample paths. In actual markets, it is reasonable to place a knockout boundary at a latter part of the trading interval for hedging risk in future. In this sense, switched knockout options with latter-part boundaries can be attractive alternatives to the vanilla option. Another marked difference is the value of each option price when $S(0) \ge B = 180$. That is, the option prices for the latter-part cases have positive values, while those for the former-part cases are always 0.

To see the effects of volatility to option prices, we compute the prices of switched knockout options with $\sigma = 0.2, 0.3, 0.4$. Figures 3 and 4 illustrate the curves of $V(S(0),0)$ as a function $S(0)$ when $\mathcal{I}_{\text{on}} = [0.5,1]$ and $\mathcal{I}_{\text{on}} = [0,0.5]$, respectively. For the vanilla option, it is well known that the price is monotonously increasing with σ, *i.e.*, $\partial V/\partial\sigma > 0$ for all $\sigma > 0$. However, we see from Figures 3 and 4 that this property does not hold for switched knockout options: Roughly speaking, for all $\sigma > 0$, $\partial V/\partial\sigma > 0$ when $S(0) << K$ and $\partial V/\partial\sigma < 0$ when $S(0) >> K$. This result indicates that a new scheme for risk hedging should be invented for switched knockout options.

References

[1] Black, F. and Scholes, M., "The pricing of options and corporate liabilities," *Journal of Political Economy*, **81** (1973) 637–654.

[2] Hanada, K. and Kimura, T., "Numerical Valuation of a Switched Knockout Option," Proceedings of ASMM'2000, Kyoto, 2000, to appear.

[3] Merton, R., "The theory of rational option pricing," *Bell Journal of Economics and Management Science*, **4** (1973) 141–183.

[4] Rich, D.R., "The mathematical foundations of barrier option-pricing theory," *Advances in Futures and Options Research*, **7** (1994) 267–311.