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Option Pricing Models in an Uncertain Environment

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1. Abstract. ファイナンスのアメリカン・オプション価格付け問題において、オプションの売買時に不確実性がある場合を考える。確率モデルにファジイ理論を導入することで、効用関数にもとづく主観的な決定基準のもとでオプション価格の許容範囲を求める方法を論じる。

2. Option Pricing Models. We consider American put option in a finance model where there is no arbitrage opportunities. $\mathbb{R}$ denotes the set of all real numbers. Let μ be the appreciation rate and let σ be the volatility ($\mu \in \mathbb{R}$, $\sigma > 0$). Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{M}, P)$. $\{\mathcal{M}_t\}_{t\geq 0}$ denotes a family of nondecreasing right-continuous complete sub-σ-fields of $\mathcal{M}$ such that $\mathcal{M}_t$ is generated by $B_s (0 \leq s \leq t)$. We consider two assets, a bond price $\{R_t\}_{t\geq 0}$ and a stock price $\{S_t\}_{t\geq 0}$, where the bond price process $\{R_t\}_{t\geq 0}$ is riskless and the stock price process $\{S_t\}_{t\geq 0}$ is risky. Let r ($r \geq 0$) be the instantaneous interest rate, i.e. interest factor, on a bond. Let a bond price process $\{R_t\}_{t\geq 0}$ be $R_t = e^{rt}, t \geq 0$. Let a stock price process $\{S_t\}_{t\geq 0}$ satisfy the log-normal stochastic differential equation: S_0 is a positive constant, and

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \quad t \geq 0. \tag{1}$$

In this paper, we present option models where a stock price process S_t takes fuzzy values using fuzzy random variables, whose mathematical notations are introduced by [2].

We deal with models with the time space $\mathbb{T} = [0, T]$ and $\mathbb{T} = [0, \infty)$, where T is a positive constant. Let $\{a_t\}_{t\geq 0}$ be an $\mathcal{M}_t$-adapted stochastic process such that the map $t \mapsto a_t(\omega)$ is continuous on $[0, \infty)$ and $0 < a_t(\omega) \leq S_t(\omega)$ for almost all $\omega \in \Omega$. We give a fuzzy stochastic process of the stock price process $\{\tilde{S}_t\}_{t\geq 0}$ by the following fuzzy random variables:

$$\tilde{S}_t(\omega)(x) := \begin{cases} L((S_t(\omega) - x)/a_t(\omega)) & \text{if } x \leq S_t(\omega) \\ L((x - S_t(\omega))/a_t(\omega)) & \text{if } x > S_t(\omega) \end{cases} \tag{2}$$

for $t \geq 0$, $\omega \in \Omega$ and $x \in \mathbb{R}$, where $L(x) := \max\{1 - |x|, 0\}$ ($x \in \mathbb{R}$) is the triangle-type shape function. Hence, $a_t(\omega)$ is a half width of triangular fuzzy numbers $\tilde{S}_t(\omega)$ and corresponds to the amount of fuzziness in the process. Let K be a strike price ($K > 0$). We define the fuzzy price process by a fuzzy stochastic process $\{\tilde{P}_t\}_{t\geq 0}$:

$$\tilde{P}_t(\omega) := e^{-rt}(1_{\{K\}} - \tilde{S}_t(\omega)) \vee 1_{\{0\}} \quad \text{for } t \geq 0,\ \omega \in \Omega, \tag{3}$$

where $\vee$ is the maximum by fuzzy max order, and $1_{\{K\}}$ and $1_{\{0\}}$ denote the crisp numbers K and zero respectively. Fix an initial stock price y ($y = S_0 > 0$). Put the optimal fuzzy price of American put option by

$$\tilde{V} = \vee_\tau\, E(\tilde{P}_\tau), \tag{4}$$

where $E(\cdot)$ denotes expectation with respect to the equivalent martingale measure Q, the α-cut of $E(\tilde{P}_\tau)$ is defined by $E(\tilde{P}_\tau)_\alpha := [\int_\Omega \tilde{P}^-_{\tau,\alpha}(\omega)\, \mathrm{d}P(\omega), \int_\Omega \tilde{P}^+_{\tau,\alpha}(\omega)\, \mathrm{d}P(\omega)]$, and $\vee$ means the supremum with respect to the fuzzy max order over stopping times τ with values in $\mathbb{T}$.

We consider a valuation method of fuzzy prices, taking into account of decision maker's subjective judgement. Give a fuzzy goal by a fuzzy set $\varphi : [0, \infty) \mapsto [0, 1]$ which is a continuous and increasing function with $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(x) = 1$. Then we note that the α-cut is $\varphi_\alpha = [\varphi^-_\alpha, \infty)$ for $\alpha \in (0, 1)$. For a stopping time τ, we define a fuzzy expectation of the fuzzy numbers $E(\tilde{P}_\tau)$ by

$$\tilde{E}(E(\tilde{P}_\tau)) := \int\!\!\!\!-_{[0,\infty)} E(\tilde{P}_\tau)(x)\ \mathrm{d}\tilde{m}(x) = \sup_{x\geq 0} \min\{E(\tilde{P}_\tau)(x), \varphi(x)\}, \tag{5}$$

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where $\tilde{m}$ is the possibility measure generated by the density φ and $⨍ \, d\tilde{m}$ denotes Sugeno integral. The fuzzy number $E(\tilde{P}_\tau)$ means a fuzzy price, and the fuzzy expectation (5) implies the degree of writer's (seller's) satisfaction regarding fuzzy prices $E(\tilde{P}_\tau)$. Then the fuzzy goal $\varphi(x)$ means a kind of utility function for expected prices x in (5), and it represents a writer's (seller's) subjective judgement from Bellman and Zadeh's idea.

Problem P. Find a stopping time τ^* with values in $\mathbb{T}$ such that $\tilde{E}(E(\tilde{P}_{\tau^*})) = \tilde{E}(\tilde{V})$.

Then, τ^* is called an optimal exercise time and a real number $x^*(\geq 0)$ is called an optimal expected price under the fuzzy expectation generated by possibility measures (see (5)) if it attains the supremum of the fuzzy expectation (5), i.e.

$$\tilde{E}(\tilde{V}) = \sup_{x \geq 0} \min\{\tilde{V}(x), \varphi(x)\} = \min\{\tilde{V}(x^*), \varphi(x^*)\}. \tag{6}$$

Fix an initial stock price $y(>0)$. In this section, we discuss the optimal expected price of American put option $\tilde{V} := \tilde{V}(y, 0)$, which is introduced in the previous section, and we give an optimal exercise time for Problem P. Define a grade α^* by $\alpha^* := \sup\{\alpha \in [0,1] | \varphi_\alpha^- \leq \tilde{V}_\alpha^+\}$, where $\varphi_\alpha = [\varphi_\alpha^-, \infty)$ for $\alpha \in (0,1)$, and the supremum of the empty set is understood to be 0.

Theorem. *Under the fuzzy expectation by possibility measures (5), the following hold.*

(i) *The grade of the fuzzy expectation of American put option price $\tilde{V}$ is given by*

$$\alpha^* = \tilde{E}(\tilde{V}) = \sup_{\tau:\text{ stopping times with values in } \mathbb{T}} \tilde{E}(E(\tilde{P}_\tau)). \tag{7}$$

(ii) *Further, the optimal expected price of American put option is given by*

$$x^* = \varphi_{\alpha^*}^-. \tag{8}$$

(iii) *If a stopping time*

$$\tau^*(\omega) := \tau_{\alpha^*}(\omega) = \inf\{t \in \mathbb{T} \mid S_t(\omega) \leq s^+(\alpha^*)\}, \quad \omega \in \Omega, \tag{9}$$

is finite, then τ^* is an optimal stopping time for Problem P.

Since the fuzzy expectation (6) is defined by possibility measures, (8) gives an upper bound on optimal expected prices of American put option. Therefore, similarly to (4.1) we can define another grade, which gives a lower bound on optimal expected prices of American put option as follows:

$$x_* = \varphi_{\alpha_*}^-, \tag{10}$$

where α_* is defined by $\alpha_* := \sup\{\alpha \in [0,1] | \varphi_\alpha^- \leq \tilde{V}_\alpha^-\}$. Its corresponding stopping time is given by

$$\tau_*(\omega) := \tau_{\alpha_*}(\omega) = \inf\{t \in \mathbb{T} \mid S_t(\omega) \leq s^-(\alpha_*)\}, \quad \omega \in \Omega. \tag{11}$$

Hence, $[x_*, x^*]$ means writer's tolerance of expected prices under his fuzzy goal φ.

Example 1. Consider American put option with an expiration date T ($\mathbb{T} = [0, T)$). Put a fuzzy goal $\varphi(x) = 1 - e^{-0.1x} (x \geq 0)$ Then, $\varphi_\alpha^- = -0.1^{-1} \log(1-\alpha)$ for $\alpha \in (0,1)$. Put an expiration date $T = 10$, a volatility $\sigma = 0.25$, an interest factor $r = 0.05$, a fuzzy factor $c = 0.05$, an initial stock price $y = 30$ and a stock price $K = 35$. We can easily calculate that the optimal grades are $\alpha_* \approx 0.969769152$ and $\alpha^* \approx 0.969769253$. These grade mean the degrees of writer's satisfaction in pricing of American put option. A corresponding writer's tolerance of expected prices under his fuzzy goal φ is $[x_*, x^*] \approx [34.988924, 34.988958]$.

References

[1] Y.Yoshida, The optimal stopped fuzzy rewards in some continuous-time dynamic fuzzy systems, *Math. and Comp. Modelling* **26** (1997) 53-66.

[2] 吉田祐治, On Optimal Stopping of a Sequence of Random Variables with Fuzziness (1998.10), 日本OR学会アブストラクト.

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