USING ARITHMETIC PROCESSES TO STUDY MAINTENANCE PROBLEMS FOR ENGINES

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In this study, first the sequential survival times and the sequential repair times of an engine were modelled using a decreasing arithmetic process and an increasing arithmetic process respectively. Secondly the values of model parameters were estimated by applying special statistical methods. Finally the means of the sequential survival times and the means of the sequential repair times of an engine were estimated, and the optimum policy for the replacement of the engine was determined.

Significance: The performance of an engine decreases because of ageing and accumulated wear. The effects of ageing and accumulated wear are assumed to be irreversible, so that after a repair an engine will not work as well as it did when new. Consequently the survival time of an engine becomes shorter and shorter and the repair time of the engine becomes longer and longer. After a certain number of failures, an engine can still work but is no longer maintainable in a cost effective way. At this epoch, the engine should be replaced with a new one.

Keywords: Arithmetic Process, Renewal Process, Repair Time, Replacement, Survival Time.

1. INTRODUCTION

The Kowloon Motor Bus (KMB) Company Limited employs different types of bus for the provision of public transport within Hong Kong. An engine is one of the main components of a bus. Every time an engine fails, KMB will repair it. Since the operation quality of an engine deteriorates under ageing and accumulated wear, the survival time decreases and the repair time increases. Eventually, it is no longer worthwhile to repair a fatigued engine and it becomes far efficient to replace it with a new one.

The perfect repair model and the minimal repair model are popular models used in resolving maintenance problems [1,2]. The perfect repair model assumes that a failed system will work as well as a new one after repair. The minimal repair model assumes that a failed system will function, after repair, with the same rate of failure and the same effective age as at the instant of the last failure. Neither model is, however, particularly relevant to the modelling of the engine maintenance problems that we encounter here because the repair time in these two models is negligible. In this study, we use two different arithmetic processes [3,4] to model the failure and repair processes of an engine.

Condition 1. Given a sequence of random variables $X_1, X_2, X_3, \ldots$, if for some real number a, $\{X_i+(i-1)a,\ i = 1,2,3,\ldots\}$ forms a renewal process, $\{X_i,\ i = 1,2,3,\ldots\}$ is an arithmetic process. a is called the common difference of the arithmetic process.

Condition 2. If $a \in \left(0, \dfrac{\mu_{X_1}}{i-1}\right]$, where $i = 2,3,4,\ldots$ and μ_{X_1} is the mean of the first random variable X_1, then the arithmetic process is called a decreasing arithmetic process. If $a < 0$, then the arithmetic process is called an increasing arithmetic process. If $a = 0$, then the arithmetic process reduces to a renewal process.

The upper bound of a in Condition 2 can be obatined as follows: By Condition 1, the expression for the general term of an arithmetic process is given by $X_i = X_1 - (i-1)a$. Taking expectations on both sides of this expression, and remembering that X_i is a non-negative random variable and hence $\mathrm{E}(X_i) \equiv \mu_{X_i} \geq 0$ for $i = 1,2,3,\ldots$; we obtain, after transposition, the upper bound of a given by $\dfrac{\mu_{X_1}}{i-1}$ for $i = 2,3,4,\ldots$.

If the sequential survival times of an engine decrease arithmetically with a common difference, we can model these times using a decreasing arithmetic process. Furthermore, if the sequential repair times of an engine increase arithmetically with a common difference, we can model them using an increasing arithmetic process.

2. OBJECTIVES

The objectives of this study are to determine the following:
(1) the common difference of the decreasing arithmetic process for the sequential survival times of each type of engine.
(2) the common difference of the increasing arithmetic process for the sequential repair times of each type of engine.
(3) the means of the sequential survival times of each type of engine.
(4) the means of the sequential repair times of each type of engine.
(5) the mean lifetime of each type of engine.
(6) the optimal replacement policy that should be adopted for each type of engine based on the minimisation of the long-run average cost per day.

3. ASSUMPTIONS

Model assumptions are as follows:
(1) Initially a new engine is used.
(2) Whenever an engine fails, we can repair it.
(3) Let X_i be the survival time after the $(i-1)$th repair. Then $\{X_i,\ i = 1,2,3,...\}$ forms a decreasing arithmetic process with a common difference $a \in \left(0, \dfrac{\mu_{X_1}}{i-1}\right]$, where $i = 2,3,...$ and μ_{X_1} is the mean of the first survival time X_1.
(4) Let Y_i be the repair time after the ith failure. Then $\{Y_i,\ i = 1,2,3,...\}$ forms an increasing arithmetic process with a common difference $b < 0$.
(5) The failure process $\{X_i,\ i = 1,2,3,...\}$ and the repair process $\{Y_i,\ i = 1,2,3,...\}$ are independent.
(6) All the engines work under nearly identical conditions.

4. METHODOLOGY

4.1 Notation for Variables

$E(X_1) \equiv \mu_{X_1} \equiv \mu_X > 0$ and $V(X_1) \equiv \sigma^2_{X_1} \equiv \sigma^2_X$ are the mean and variance of X_1 respectively.

$E(Y_1) \equiv \mu_{Y_1} \equiv \mu_Y \geq 0$ ($\mu_Y = 0$ means that the repair time is negligible) and $V(Y_1) \equiv \sigma^2_{Y_1} \equiv \sigma^2_Y$ are the mean and variance of Y_1 respectively.

c_r is the average cost of a repair per day.

c_p is the average cost of a replacement.

4.2 Testing for an Arithmetic Process

4.2.1 Testing the Existence of a Trend in the Data

For ease of manipulation and interpretation, the Laplace test is used [1].

Null hypothesis, H_0 : X_i's are identically distributed.

Alternative hypothesis, H_1 : X_i's are not identically distributed, i.e. there is a trend.

The test statistic :

$$U = \left[\frac{\sum_{j=1}^{n-1} T_j}{n-1} - \frac{T_n}{2}\right] \Bigg/ \left[T_n \sqrt{\frac{1}{12(n-1)}}\right] \quad \ldots \quad (1)$$

where $T_j = \sum_{i=1}^{j} X_i$

is approximately distributed as the standard normal for $n \geq 3$ at the 0.05 level of significance.

The decision rule : Reject H_0 if $U > 1.96$ or $U < -1.96$, i.e. the data set exhibits an upward trend or a downward trend respectively.

4.2.2 Testing Whether the Data Come from an Arithmetic Process [4]

First, we plot X_i against $(i-1)$ to see whether there is a linear relationship between them. If so, a simple linear regression model can be written as follows:

$$X_i = -a(i-1) + \alpha + \varepsilon_i \qquad \cdots \qquad (2)$$

where $E(\varepsilon_i) = 0$ and $V(\varepsilon_i) \equiv \sigma_\varepsilon^2$.

Secondly, we estimate the parameters a and α using the least-square criterion. The least-square estimators for a and α are as follows:

$$\hat{a} = \frac{2}{(n-1)n(n+1)}\left[3(n-1)\sum_{i=1}^{n} X_i - 6\sum_{i=1}^{n}(i-1)X_i\right] \qquad \cdots \qquad (3)$$

and

$$\hat{\alpha} = \frac{2}{n(n+1)}\left[(2n-1)\sum_{i=1}^{n} X_i - 3\sum_{i=1}^{n}(i-1)X_i\right] \qquad \cdots \qquad (4)$$

Thirdly, we calculate the mean square error σ_ε^2 for the simple linear regression model using the following estimator.

$$\hat{\sigma}_\varepsilon^2 = \frac{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2 - \hat{a}\left[\frac{(n-1)}{2}\sum_{i=1}^{n} X_i - \sum_{i=1}^{n}(i-1)X_i\right]}{n-2} \qquad \cdots \qquad (5)$$

Finally, we distinguish a renewal process from an arithmetic process using the following hypothesis testing at the 0.05 level of significance.

Null hypothesis, $H_0 : a = 0$.

Alternative hypothesis, $H_1 : a \neq 0$.

The test statistic: $t = \dfrac{-\hat{a}\sqrt{(n-1)n(n+1)}}{\sqrt{12}\hat{\sigma}_\varepsilon}$ $\qquad \cdots \qquad (6)$

is approximately distributed as a student t with $(n-2)$ degrees of freedom.

The decision rule: Reject H_0 if $|t| >$ the critical value.

We can apply the same procedure, i.e. use equations (1) to (6) to a set of sequential repair times $\{Y_i,\ i = 1,2,3,...\}$ to check whether the data come from an arithmetic process.

4.3 Estimating the Means and Variances of X_1 and Y_1

$E(X_1)$, $V(X_1)$, $E(Y_1)$ and $V(Y_1)$ are determined by estimating the parameters μ_X, σ_X^2, μ_Y and σ_Y^2 respectively [4]. These parameters are estimated using the relevant estimators listed in Table 1 [5].

Table 1. Estimators for μ_X, σ_X^2, μ_Y and σ_Y^2

	a		b	
	$\hat{\mu}_X$	$\hat{\sigma}_X^2$	$\hat{\mu}_Y$	$\hat{\sigma}_Y^2$
$=0$	$\frac{\sum_{i=1}^{n} X_i}{n} \equiv \bar{X}$	$\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}$	$\frac{\sum_{i=1}^{n} Y_i}{n} \equiv \bar{Y}$	$\frac{\sum_{i=1}^{n}(Y_i-\bar{Y})^2}{n-1}$
<0	$\hat{\alpha}$	$\frac{\sum_{i=1}^{n}\hat{Z}_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n}\hat{Z}_i\right)^2}{n-1}$ where $\hat{Z}_i = X_i + (i-1)\hat{a}$	$\hat{\alpha}$	$\frac{\sum_{i=1}^{n}\hat{Z}_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n}\hat{Z}_i\right)^2}{n-1}$ where $\hat{Z}_i = Y_i + (i-1)\hat{b}$
>0	$\ln\left(\frac{\hat{\mu}_X}{\hat{\alpha}}\right) - \frac{\hat{\sigma}_\varepsilon^2}{2}\left(\frac{1}{\hat{\mu}_X^2} - \frac{1}{\hat{\alpha}^2}\right) = 0$	As above	$\ln\left(\frac{\hat{\mu}_Y}{\hat{\alpha}}\right) - \frac{\hat{\sigma}_\varepsilon^2}{2}\left(\frac{1}{\hat{\mu}_Y^2} - \frac{1}{\hat{\alpha}^2}\right) = 0$	As above

4.4 Determining the Means and Variances of X_i and Y_i

The mean and variance of the survival time and the repair time of each type of engine are estimated respectively using the following equations.

$$\hat{\mu}_{X_i} = \hat{\mu}_X - (i-1)\hat{a},\ \hat{\sigma}_{X_i}^2 = \hat{\sigma}_X^2,\ \hat{\mu}_{Y_i} = \hat{\mu}_Y - (i-1)\hat{b} \text{ and } \hat{\sigma}_{Y_i}^2 = \hat{\sigma}_Y^2 \qquad \cdots \qquad (7)$$

4.5 Determining the Optimum Replacement Policy

The optimum replacement policy is determined by minimising the (expected) long-run average cost per unit time.

$$\text{The (expected) long-run average cost per unit time} = \frac{\text{the expected cost incured in a cycle}}{\text{the expected length of a cycle}}$$

where a cycle is the time between two successive replacements.

The following steps show how to determine the optimum replacement policy [3].

(1) Calculate the long-run average cost per day $c(i)$ for i=1,2,3,... using the following equation.

$$c(i) = \frac{c_r\left\{\frac{i-1}{2}\left[2\mu_Y - (i-2)b\right]\right\} + c_p}{\frac{i}{2}\left[2\mu_X - (i-1)a\right] + \frac{i-1}{2}\left[2\mu_Y - (i-2)b\right]} \quad \cdots \quad (8)$$

(2) Plot the long-run average cost per day $c(i)$ against i. Then the optimum replacement epoch, shown by the minimum point of the curve, is determined.

5. ANALYSIS AND FINDINGS

The Depot Manager provided us with a total of 1,503 repair records for three different types of engine, namely engine types: 407H, 6LXB, 6LXCT. Table 2 shows an example of the raw repair records. We sorted the repair records by engine type and next by engine number. In addition, we only analysed those engines which had more than one repair record.

Table 2. An Example of the Raw Repair Records

BUS_NO	ENG_NO	E_TYPE	V_TYPE	DEPOT	R_DATE	RET_DATE	INS_DATE
DG2612	028879	407H	BENZ	TM	8/1/93	7/30/93	7/19/93
DG5362	028849	407H	BENZ	TM	8/9/93	8/1/93	10/23/89
DF8962	046787	407H	BENZ	TM	8/28/93	8/1/93	11/23/89
DF6548	029344	407H	BENZ	TM	4/1/94	3/28/94	6/9/90
DG2612	028827	407H	BENZ	TM	4/9/94	4/1/94	7/31/93
DF9740	028920	407H	BENZ	TM	4/21/94	4/1/94	9/21/90
DG4756	028900	407H	BENZ	TM	6/8/94	6/1/94	10/11/90
DF9705	028901	407H	BENZ	TM	9/3/94	9/1/94	10/10/89
DF6700	028874	407H	BENZ	TM	9/21/94	9/1/94	12/6/88

The INS_DATE is when the repaired engine reinstalled in a bus. The RET_DATE is when the damaged engine is transported to the maintenance depot. The R_DATE is when the engine repair is completed. Each survival time (X_i) is determined by the difference between the INS_DATE and the RET_DATE. Each repair time (Y_i) is determined by the difference between the RET_DATE and the R_DATE.

Since calculation of the parameter values for the three different types of engine is time-consuming, equations (1) to (8) and the relevant estimators listed in Table 1 were computed using EXCEL.

We verified that the sequential survival times and the sequential repair times come from two different arithmetic processes. The mean survival time and the mean repair time were estimated, and are included in Table 3. The mean survival time, the mean repair time and the long-run average cost per day versus the number of failures are plotted in Figures 1, 2 and 3 respectively.

Table 3. The Mean Survival Time and the Mean Repair Time

Engine Type	$\hat{\mu}_{X_i} = \hat{\mu}_X - (i-1)\hat{a}$ (days)	$\hat{\mu}_{Y_i} = \hat{\mu}_Y - (i-1)\hat{b}$ (days)
407H	$1426.07 - (i-1)730$	$14.11 - (i-1)(-0.33)$
6LXB	$1241.32 - (i-1)584$	$8.86 - (i-1)(-12.17)$
6LXCT	$1629.64 - (i-1)752$	$39.03 - (i-1)(-6.13)$

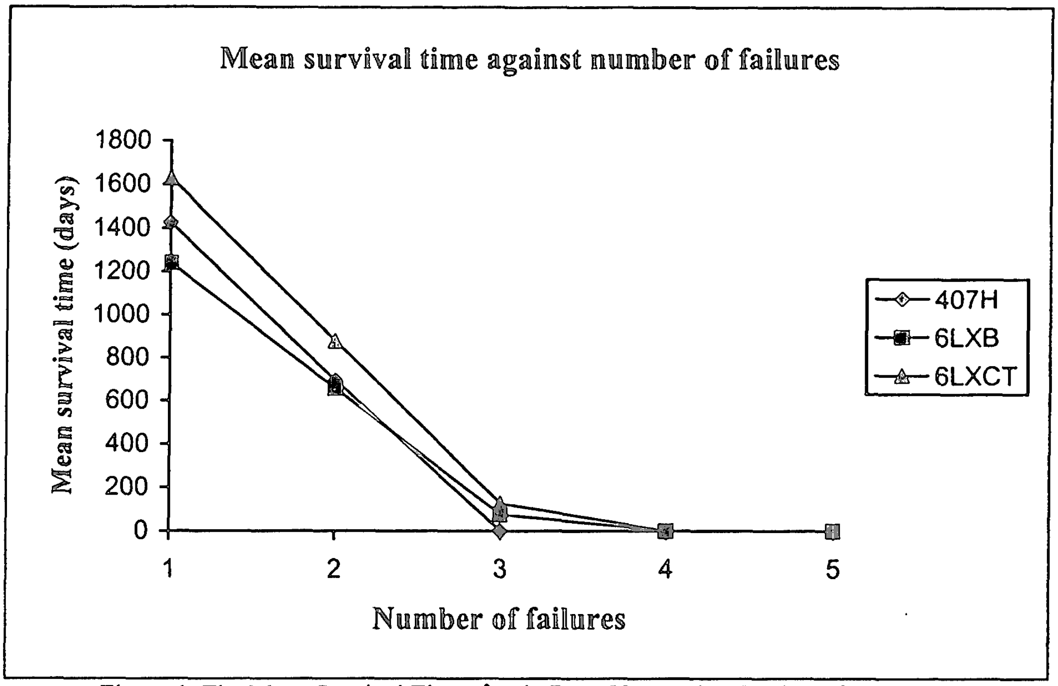

Figure 1. The Mean Survival Time $\hat{\mu}_{X_i}$ in Days Versus the Number of Failures i

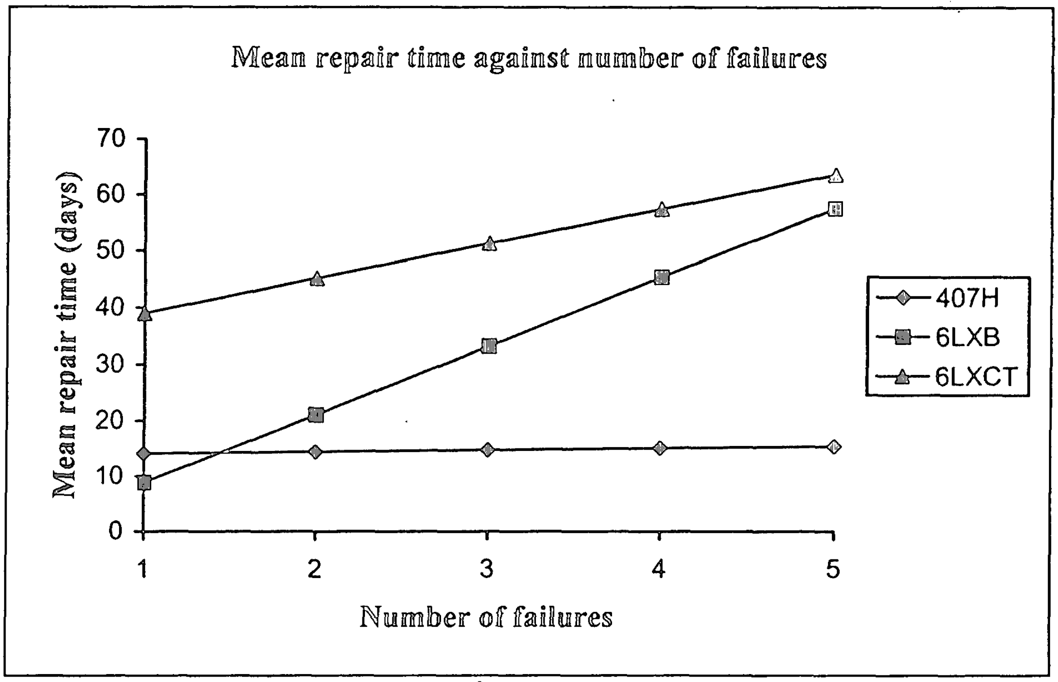

Figure 2. The Mean Repair Time $\hat{\mu}_{Y_i}$ in Days Versus the Number of Failures i

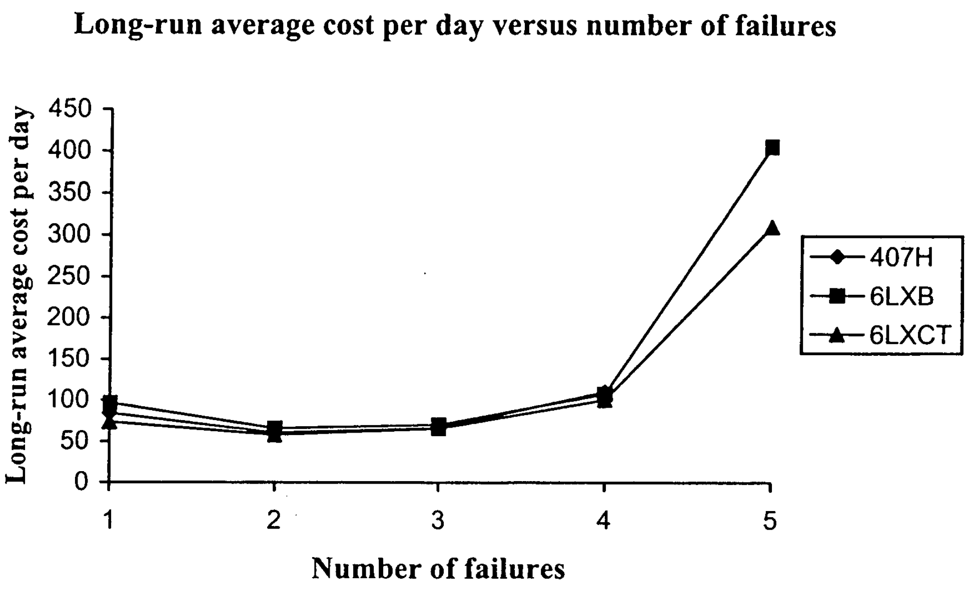

Figure 3. The Long-run Average Cost Per Day $c(i)$ Versus the Number of Failures i

6. CONCLUSION

A summary of the parameter values for each of the three different types of engine is shown in Table 4.

Table 4. Parameter Values

	Survival Times		Repair Times	
Engine Type	$\hat{a}$	$\hat{\mu}_X$ (years)	$\hat{b}$	$\hat{\mu}_Y$ (days)
407H	2.00	3.91	−0.33	14.11
6LXB	1.60	3.40	−12.17	8.86
6LXCT	2.06	4.46	−6.13	39.03

From Table 4 we can see that all the values of a are greater than zero. This implies that the sequential survival times of each type of engine form a decreasing arithmetic process. The study's findings show that an engine deteriorates over time in general. An engine's lifetime is finite because it cannot be constantly repaired so that it can function forever. From Table 4 we can also see that all the values of b are smaller than zero. This implies that the sequential repair times of each type of engine form an increasing arithmetic process. Since the effects of ageing and wear become more serious as the number of failures increases, the time needed to make repairs increases correspondingly. The repair time becomes longer and longer. The engine thus becomes unrepairable.

Table 5 shows the mean lifetimes of the three different types of engine, and indicates that the 6LXCT engine type has the longest mean lifetime.

Table 5. The Mean Lifetimes

Engine Type	Mean Lifetime $\approx \sum_{i=1}^{2}\left[\hat{\mu}_X - (i-1)\hat{a}\right]$ (years)
407H	5.82
6LXB	5.20
6LXCT	6.86

Table 6 shows the optimum replacement policies for the three different types of engine.

Table 6. The Optimum Replacement Policies Based on the Minimum Cost

Engine Type	Optimal Replacement Policy Based on Minimum Cost
407H	Replace after the 2nd failure
6LXB	Replace after the 2nd failure
6LXCT	Replace after the 2nd failure

Although Table 6 shows that almost all optimum replacement epoches come after the 2nd failure, the survival time of an engine after the 2nd failure is very short. As such, it is neither practical nor cost effective to continue to make repairs after this point.

The parameter values of a and μ_X are highly over- and under-estimated respectively owing, in part, to the violation of the assumption that all the engines operate under identical environments. Since different engine types are installed in different buses and are used for different routes, each engine operates in a significantly different environment. The performance of each engine is therefore affected by this factor.

From previous meetings with the Depot Manager, we know that the mean lifetimes of the three types of engine are far longer than the findings indicated in Table 5. The difference is due to data shortage, as KMB only keeps repair records for a few years. The consequence of this is that the value of μ_X is under-estimated, and the values of a, b and μ_Y are over-estimated. As a result, the mean lifetime of each type of engine was further under-estimated.

Although the findings cannot fully reflect the operating characteristics of the engines, we can regard this application as a pilot study on the modelling of engine maintenance problems using arithmetic processes.

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