An Adjusted Projection in DEA

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1. Introduction

DEA identifies the efficient frontiers spanned by efficient DMUs (decision making units). For an inefficient DMU, these frontiers operate as a benchmark, and inefficient DMU can be improved in efficiency by projecting onto the efficient frontiers. However, this operation needs care. Especially when we compare the performance of DMUs operating under different conditions, a straightforward comparison may cause problems of rationality and hence the induced projections will result in impractical proposals for improvement. As an example, we observe three categories of public libraries operating under different environments, i.e., those in business, commercial and residential districts. Usually, libraries in residential districts outperform others (in business and commercial districts). Hence, comparing public libraries from scratch might be unfair to libraries in the latter categories. Similar situations may arise in the case of comparisons of hotels, restaurants, supermarkets, hospitals etc., which are operating countrywide or worldwide.

Since the classical work by Charnes, Cooper and Rhodes many authors have analyzed these situations. These papers can be classified into:

1. Papers that measure efficiencies of categorized DMUs and find difference between categories by using some statistical tests.
2. Papers that assume a hierarchical structure among categories and evaluate the efficiency of DMUs in a certain category referring to DMUs in the same or less advantageous category classes.

Most of them put emphasis on the measurement of efficiency and relatively few addressed projection methodology under these situations. This may reflect the view that projection/improvement belongs to managerial decision and is not within the scope of DEA. This might be partially true but we need to develop some systematic way to deal with this problem as an aid for management.

Typical improvements by input (output) oriented DEA models consist of a radial reduction (enlargement) of inputs (outputs) plus deletion of input excesses (output shortfalls). Considering the above mentioned subjects, this paper concentrates on projection of inefficient DMUs under different business environments and proposes an "adjusted projection." The basic idea underlying this method is to think "globally" and to act "locally." First, we evaluate efficiency of each DMU with respect to all DMUs in the problem. Hence, big differences in efficiency may be found between DMUs in different categories. Then we select a "champion" DMU from each category. Using the efficiency of this champion DMU, we do an "adjusted projection" of DMUs in the same category. It will be demonstrated that the projected DMUs in a category exhibit the same level of efficiency when evaluated with respect to all DMUs in the problem. Furthermore, the projected DMUs have full efficiency when evaluated within the category. Thus, it may safely be said that we evaluate DMUs "globally" and project them "locally" still accounting and reflecting the global standard.

Recent management tends to aim at attaining a "global optimum." However, globalization is not always possible, since there exist many local constraints which restrict this movement. This paper may present a solution to this problem.

2. Methodology

We will deal with the input-oriented CCR model, although this methodology can be applied to the output-oriented case as well and to other returns-to-scale models, i.e., variable, increasing and decreasing ones.

2.1. Notation

We denote inputs and outputs of a DMU (decision making unit) by $x = (x_1, \ldots, x_m)^T$ and $y = (y_1, \ldots, y_s)^T$, respectively, where m and s are the numbers of inputs and outputs, and the symbol ^{T} designates transposition. We assume that the DMUs concerned are classified into several groups depending on their characteristics, e.g., environmental conditions, type of business etc. The total set of DMUs is presented by T and the production possibility set P spanned by T is defined by

$$P = \left\{ (x, y) \mid x \geq \sum_{j \in T} x_j \lambda_j,\ y \leq \sum_{j \in T} y_j \lambda_j \right\}, \quad (1)$$

where $\lambda_j \geq 0$ $(\forall j \in T)$ and some other constraints will be imposed on $\boldsymbol{\lambda}$ occasionally. We assume that the data set is nonnegative, i.e., $X = (\boldsymbol{x}_j) \geq \mathbf{0}$ and $Y = (\boldsymbol{y}_j) \geq \mathbf{0}$.

2.2. Adjusted Projection

The input oriented CCR model for evaluating the efficiency of a DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is described by the following linear programming (LP) problem:

$$\min \ \theta \qquad (2)$$

$$\text{subject to} \quad \theta \boldsymbol{x}_o = \sum_{j \in T} \lambda_j \boldsymbol{x}_j + \boldsymbol{s}^-$$

$$\boldsymbol{y}_o = \sum_{j \in T} \lambda_j \boldsymbol{y}_j - \boldsymbol{s}^+$$

$$\boldsymbol{\lambda} \geq \mathbf{0}, \ \boldsymbol{s}^- \geq \mathbf{0}, \ \boldsymbol{s}^+ \geq \mathbf{0}.$$

The CCR model is solved by the two phase process in DEA literature, i.e. minimize θ in the first phase and then maximize the sum of slacks in the second phase. Let an optimal solution of this problem be $(\boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$.

Definition 1 (Radial efficient) *DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is radial efficient if $\theta^* = 1$ holds.*

Definition 2 (Slackless) *DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is slackless if $\boldsymbol{s}^{-*} = \mathbf{0}$ and $\boldsymbol{s}^{+*} = \mathbf{0}$ hold.*

Definition 3 (CCR-efficient) *DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-efficient, if it is radial efficient and slackless.*

The *CCR-projection* $(\bar{\boldsymbol{x}}_o, \bar{\boldsymbol{y}}_o)$ is defined by

$$\bar{\boldsymbol{x}}_o = \theta^* \boldsymbol{x}_o - \boldsymbol{s}^{-*} \qquad (3)$$

$$\bar{\boldsymbol{y}}_o = \boldsymbol{y}_o + \boldsymbol{s}^{+*}.$$

It can be demonstrated that $(\bar{\boldsymbol{x}}_o, \bar{\boldsymbol{y}}_o)$ is CCR-efficient.

Lemma 1 (Adjusted projection) *Let us define*

$$\widetilde{\boldsymbol{x}}_o = \bar{\boldsymbol{x}}_o / \alpha \text{ and } \widetilde{\boldsymbol{y}}_o = \bar{\boldsymbol{y}}_o, \qquad (4)$$

where α is a positive scalar not greater than one $(0 < \alpha \leq 1)$. Then the radial efficiency of $(\widetilde{\boldsymbol{x}}_o, \widetilde{\boldsymbol{y}}_o)$ is α and it is slackless.

2.3. DMUs in a Group Category

We tend to make evaluations and projections of DMUs in a group category while reflecting evaluation under the total production possibility set P. Let the group of interest be denoted by A. We will follow the steps below:

Step 1 We first evaluate the efficiency of DMUs in Group A with respect to the total production possibility set P. Let the optimal solution for $A_j = (\boldsymbol{x}_j, \boldsymbol{y}_j)$ $(j \in A)$ be $(\theta_j^*, \boldsymbol{\lambda}_j^*, \boldsymbol{s}_j^{-*}, \boldsymbol{s}_j^{+*})$. We project A_j onto the efficient frontier of P and denote it by $\bar{A}_j = (\bar{\boldsymbol{x}}_j, \bar{\boldsymbol{y}}_j)$. Thus,

$$\bar{\boldsymbol{x}}_j = \theta_j^* \boldsymbol{x}_j - \boldsymbol{s}_j^{-*} \qquad (5)$$

$$\bar{\boldsymbol{y}}_j = \boldsymbol{y}_j + \boldsymbol{s}_j^{+*}.$$

Step 2 Find the maximum θ^*_{max} of θ_j^* $(j \in A)$, i.e.,

$$\theta^*_{max} = \max\{\theta_j^* | j \in A\}. \qquad (6)$$

Step 3 Define the projection $\widetilde{A}_j = (\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ of A_j within Group A by the following formula:

$$\widetilde{\boldsymbol{x}}_j = \bar{\boldsymbol{x}}_j / \theta^*_{max}, \quad \widetilde{\boldsymbol{y}}_j = \bar{\boldsymbol{y}}_j. \qquad (7)$$

The thus obtained projection $(\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ has the following characteristics:

Theorem 1 *$(\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ $(j \in A)$ has the radial efficiency equal to θ^*_{max} and is slackless.*

Theorem 2 *If we evaluate the efficiency of $\widetilde{A}_j = (\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ within Group $\widetilde{A}$, then $\widetilde{A}_j$ is CCR-efficient.*

So far, we have discussed efficiency issues mainly with respect to the original production possibility set P define by (1). Another production possibility set can be defined based on the projected DMUs $(\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ $(j \in T)$ as

$$\widetilde{P} = \left\{ (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}) \mid \widetilde{\boldsymbol{x}} \geq \sum_{j \in T} \widetilde{\boldsymbol{x}}_j \lambda_j, \ \widetilde{\boldsymbol{y}} \leq \sum_{j \in T} \widetilde{\boldsymbol{y}}_j \lambda_j \right\}. \quad (8)$$

Between P and $\widetilde{P}$ we have a relationship:

Theorem 3 $P = \widetilde{P}$.

Corollary 1 *The efficiency evaluation with respect to the data set (X, Y) is equivalent to that by the data set $(\widetilde{X}, \widetilde{Y})$.*

References

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