AN EXPLICIT FORMULA FOR THE LIMITING OPTIMAL VALUE IN THE FULL INFORMATION DURATION PROBLEM

Karelian Research Center of Russian Academy of Science　Vladimir. V. Mazalov
01303783　愛知大学　*玉置 光司　TAMAKI Mitsushi

1. Introduction

We consider here a full-information model for the duration problem (See Ferguson, Hardwik, Tamaki) with horizon n tending to infinity. Our objective here is to determine the asymptotics for the optimal value $V(n)$.

Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables, uniformly distributed on $[0,1]$, where X_n denotes the value of the object at the n-th stage from the end. We call an object relatively best if it possesses the largest value than previous objects. The task is to select a relatevily best object with the view of maximizing the duration it stays relatively best. Let $v(x,n)$ denote the optimal expected return when there are n objects yet to be observed and the present maximum of past observations is x. Notice that $V(n) = v(0,n)$.

The Optimality Equation for $v(x,n)$ has form

$$v(x,n) = xv(x,n-1) + \int_x^1 \max\{w(t,n), v(t,n-1)\}dt,$$

$$(n = 1,2,\cdots, \quad v(x,0) = 0)$$

where

$$w(x,n) = (1-x^n)/(1-x).$$

denotes the expected payoff given that the nth object from the last is a relatively best object of value $X_n = x$ and we select it.

Denote the point of intersection of functions $v(x,n-1)$ and $w(x,n)$ as x_n. It exists and unique because $w(x,n)$ are increasing in x while $v(x,n)$ are nonincreasing in x, and $w(0,n) = 1 \le \int_0^1 w(t,n) = v(0,n)$, and $w(1,n) = n > 0 = v(1,n)$.

If we stop with a relatively-best object $X_n = x$, we receive $w(x,n)$. If we continue and select the next relatively-best object, we expect to receive

$$\begin{aligned} u(x,n) &= \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t,n-k)dx \\ &= \sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k} (1-x^j)/j. \end{aligned}$$

It is easy to see that $u(x,n)$ satisfies the following relation.

$$u(x,n) = xu(x,n-1) + \int_x^1 w(t,n-1)dt, \quad u(x,1) = 0.$$

The problem is monotone [Ferguson et al, 1992], so the one-stage look-ahead rule (OLA) is optimal and prescribes stopping if $w(x,n) \ge u(x,n)$; that is, if

$$\sum_{k=1}^{n-1} x^{k-1} \left(1 - \sum_{j=1}^{n-k} (1-x^j)/j\right) \ge 0.$$

Equivalently we stop on step n if the relatively-best object has value $X_n \ge x_n$.

z_n written as $x_n = 1 - z_n/n$ satisfies the equation

$$\sum_{k=1}^{n-1} (1-\frac{z_n}{n})^{k-1} \left(1 - \sum_{j=1}^{n-k} (1-(1-\frac{z_n}{n})^j)/j\right) = 0,$$

and from here z_n must converge to a constant, $z_n \to z$, where $z \approx 2.11982$ satisfies the integral equation

$$\int_0^1 e^{-zv}\left[1 - \int_0^{1-v} (1-e^{-zu})/udu\right]dv = 0.$$

(See also Porosinski.)

2. Limiting optimal value

Let us introduce two new functions

$$\begin{aligned} y(x,n) &= v(x,n) - u(x,n+1), \\ \Delta_n(x) &= u(x,n) - w(x,n). \end{aligned}$$

In the interval $[0, x_n]$ both functions are non-negative and $\Delta(x_n, n) = 0$. It is easy to see that $y(x, n)$ satisfies the equation

$$y(x,n) = xy(x,n-1) + \int_x^{x_n} \Big[y(t,n-1) + \Delta_n(t)\Big] dt, \quad 0 \le x \le x_n,$$

and $y(x,n) = 0$, for $x \ge x_n$. Also, notice that $y(x,1) = y(x,2) = 0$ (because $x_1 = x_2 = 0$) and $y(x,3) = \int_x^{x_3} \Delta_3(t)dt$, where $\Delta_3(x) = 1/2 - x - (5/2)x^2$.

Now we have the following lemmas

Lemma 1. *Function $\Delta_n(x)$ satisfies the equations*

$$\Delta_{n+1}(x) - \Delta_n(x) = \sum_{j=1}^{n} \frac{x^{n-j} - x^n}{j} - x^n.$$

$$(n = 2, 3, \cdots, j \quad \Delta_1(x) = -1)$$

Lemma 2. *$y(x,n)$ satisfies the equations*

$$y(x,n) = \sum_{j=i}^{n} \int_x^{x_j} t^{n-j} \Delta_j(t) dt,$$

$$x_{i-1} \le x \le x_i, i = 3, 4, \cdots, n.$$

Lemma 3.

$$y_n = y(0,n) = \sum_{j=3}^{n} \int_0^{x_j} t^{n-j} \Delta_j(t) dt.$$

Consider the difference $r_n = y_{n+1} - y_n$. We can represent it as a sum of two expressions

$$\begin{aligned} r_n &= y_{n+1} - y_n \\ &= \sum_{j=3}^{n} \int_0^{x_j} t^{n-j} [\Delta_{j+1}(t) - \Delta_j(t)] dt \\ &+ \sum_{j=3}^{n+1} \int_{x_{j-1}}^{x_j} t^{n-j+1} \Delta_j(t) dt. \end{aligned}$$

The first sum can be rewritten in the form

$$\begin{aligned} &\sum_{j=3}^{n} \int_0^{x_j} t^{n-j} \left[\sum_{i=1}^{j} \frac{t^{j-i} - t^j}{i} - t^j \right] \\ &= \sum_{j=3}^{n} \left\{ \sum_{i=1}^{j} \left[\frac{x_j^{n-i+1}}{n-i+1} - \frac{x_j^{n+1}}{n+1} \right] \frac{1}{i} - \frac{x_j^{n+1}}{n+1} \right\}. \end{aligned}$$

As $n \to \infty$ for $x_n = 1 - z_n/n$ we have that this converges to the integral

$$\begin{aligned} V^* &= \int_0^1 e^{-\frac{z}{u}} \left[\int_0^u dv \left(\frac{e^{\frac{zv}{u}} - 1}{v} + \frac{e^{\frac{zv}{u}}}{1-v} \right) dv - 1 \right] du \\ &\approx 0.435178. \end{aligned}$$

The second sum can be shown to tends to zero as $n \to \infty$.

Theorem 1. *For large n*

$$\frac{V_n}{n} \to V^*,$$

3. PPP approach

Samuels considered our problem in PPP (Planar Poisson process) approach. He showed that the optimal limiting policy of the duration problem has $c/(1-t)$ threshold-rule and that the limiting duration under $c/(1-t)$ threshold-rule can be calculated as

$$\begin{aligned} U^* &= \int_0^1 \int_0^t E\left[D\left(s, \frac{c}{1-s}\right)\right] f_S(s) f_T(t) ds dt \\ &+ \int_0^1 \int_0^s \left\{ \int_0^{\frac{c}{1-t}} E[D(t,y)] \frac{1-t}{c} dy \right\} f_T(t) f_S(s) dt ds \end{aligned}$$

where

$$\begin{aligned} E[D(t,y)] &= \frac{1 - e^{-y(1-t)}}{y}, \\ f_T(t) &= c(1-t)^{c-1}, \\ f_S(s) &= \frac{cs}{(1-s)^{c+2}} e^{-\frac{cs}{1-s}}. \end{aligned}$$

Straightforward calculations from these immediately yield

$$U^* = \{I(c) - 1 + e^{-c}\} + \{(1+c)(e^c - 1) - ce^c I(c)\} J(c),$$

where

$$\begin{aligned} I(c) &= \int_0^1 \frac{1 - e^{-cu}}{u} du \\ J(c) &= \int_1^{\infty} \frac{e^{-cv}}{v} dv. \end{aligned}$$

It is not diffecult to show that for $c = z$, U^* agrees with V^*

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