Optimal Design of Unreliable Production/Inventory Systems with Variable Production Rate

Bibhas Chandra Giri, Tadashi Dohi (01307065) and Won Young Yun*

Department of Information Engineering, Hiroshima University, Japan
*Department of Industrial Engineering, Pusan National University, Korea

1. Summary

Khouja and Mehrez [1] considered a flexible production rate in the classical economic production lot size (EPL) model. They assumed that the process may shift from an "in-control" state to an "out-of-control" state at any random time where it starts producing no-conforming items. They formulated the model to determine the production rate and production lot size that minimize the annual expected total cost. In this article, we look at the problem of Khouja and Mehrez [1] as an economic manufacturing quantity (EMQ) problem with stochastic machine breakdown and repairs. We first formulate the model with general failure and repair time distributions and then derive the optimal lot size and optimal production rate under exponential machine failure and repair time distributions. We also extend the model to the case where certain safety stocks in inventory may be useful to improve service level to customers.

2. Assumptions and notation

In a single-machine single-product situation, the machine is subject to stochastic breakdown and repairs. When the machine failure occurs, it is detected immediately and then maintenance activity is started. After maintenance, the machine is restored back to the same initial working conditions. The demand rate is a known constant whereas the production rate is a decision variable. Both the failure rate and unit production cost depend on the production rate. Upon machine failure, the demand is met first from the on-hand inventory accumulated during the production run. If the on-hand inventory is sufficient to meet up demand during machine repair, then a new production cycle is not started until the inventory level comes down to the zero level. Shortages may occur due to longer repair time in the case where all the unsatisfied demands are lost.

To develop the model we use the following notation:
D - constant demand rate, P - variable production rate, C_0 - setup cost, C_h - holding cost, C_l - opportunity cost due to lost sales, C_r - corrective repair cost, $C(P)$ - unit production cost, T - r.v. denoting time to machine failure, S - r.v. denoting repair time upon machine failure, $f(\cdot), F(\cdot)$ - *p.d.f., c.d.f.* of T, $g(\cdot), G(\cdot)$ - *p.d.f., c.d.f.* of S, Q - production lot size.

3. Model formulation

Expected cycle length of the production-inventory system is given by

$$W_1(P,Q) = \int_0^\infty E\,[\text{duration of a cycle} \mid T = t]\, f(t)\, dt$$
$$= \int_0^{Q/P} \Big[\int_0^{(P-D)t/D} \frac{Pt}{D} g(s)\, ds + \int_{(P-D)t/D}^\infty (t+s) g(s) ds \Big] dF(t) + \int_{Q/P}^\infty \frac{Q}{D} dF(t). \quad (1)$$

Expected total cost per cycle is

$$U_1(P,Q) = \int_0^\infty E\,[\text{cost per cycle} \mid T = t]\, f(t)\, dt$$
$$= \int_0^{Q/P} \Big[\int_0^{(P-D)t/D} \Big\{ C_0 + \frac{C_h (P-D) P t^2}{2D} + C(P) Pt + C_r s \Big\} g(s)\, ds + \int_{(P-D)t/D}^\infty \Big\{ C_0 + \frac{C_h (P-D)Pt^2}{2D} + C(P)Pt + C_r s + C_l (sD - (P-D)t) \Big\} g(s)\, ds \Big] dF(t)$$
$$+ \int_{Q/P}^\infty \Big\{ C_0 + \frac{C_h (P-D) Q^2}{2PD} + C(P) Q \Big\} dF(t). \quad (2)$$

Our objective is to find the optimal production rate P^* and optimal lot size Q^* that minimize

$$ETC_1(P,Q) = U_1(P,Q) / W_1(P,Q)$$

subject to

$$D < P \quad (3)$$
$$P \le P_{\max}, \quad (4)$$

where $P_{\max}$ is the designed maximum production rate.

4. Model under exponential failure and repair time distributions

Let $F(t;P) = 1 - \exp\{-\lambda(P)t\}$, where $\lambda(P) = f(t;P)/(1-F(t;P))$ and $G(s) = 1 - \exp\{-\mu s\}$, $\mu\,(>0)$ being the repair rate upon machine failure. Integrating and simplifying equations (1) and (2), we get

$$W_1(P,Q) = \frac{P}{D\lambda(P)} \{ 1 - e^{-\lambda(P)Q/P} \}$$

$$+\frac{\lambda(P)}{\mu\left\{\lambda(P)+\frac{\mu(P-D)}{D}\right\}}\left[1-e^{-\left\{\lambda(P)+\frac{\mu(P-D)}{D}\right\}\frac{Q}{P}}\right], \quad (5)$$

$$U_1(P,Q)=C_0+\left\{\frac{PC(P)}{\lambda(P)}+\frac{C_r}{\mu}\right\}\left\{1-e^{-\lambda(P)Q/P}\right\}$$

$$+C_h(P-D)\frac{P}{D}\left[\frac{1-e^{-\lambda(P)Q/P}}{\{\lambda(P)\}^2}-\frac{Q}{P\lambda(P)}e^{-\lambda(P)Q/P}\right]$$

$$+\frac{C_l D\lambda(P)}{\mu\left\{\lambda(P)+\frac{\mu(P-D)}{D}\right\}}\left[1-e^{-\left\{\lambda(P)+\frac{\mu(P-D)}{D}\right\}\frac{Q}{P}}\right]. \quad (6)$$

Proposition 1. For any given $P(D<P\le P_{max})$, $\lambda(P)$ and $C(P)$, $W_1(P,Q)$ is concave for all $Q>0$.

Proposition 2. Let $ETC_1(P,Q_2)\le ETC_1(P,Q_1)$ for two distinct Q_1 and Q_2 of Q, given $P(D<P\le P_{max})$, $\lambda(P)$ and $C(P)$. Then $ETC_1(P,Q)$ is a pseudoconvex function provided that $U(P,Q_2)\ge U(P,Q_1)+(Q_2-Q_1)\frac{\partial U_1}{\partial Q}|_{Q=Q_1}$.

If ϕ and ψ are the Lagrange multipliers corresponding to the constrained (3) and (4) respectively, then the Kuhn-Tucker (KT) necessary conditions for optimality give

$$W_1(P,Q)\frac{\partial U_1(P,Q)}{\partial Q}-U_1(P,Q)\frac{\partial W_1(P,Q)}{\partial Q}=0 \quad (7)$$

$$W_1(P,Q)\frac{\partial U_1(P,Q)}{\partial P}-U_1(P,Q)\frac{\partial W_1(P,Q)}{\partial P}$$

$$-(\phi-\psi)W_1^2(P,Q)=0 \quad (8)$$

$$\phi(P-D)=0 \quad (9)$$

$$\psi(P_{max}-P)=0 \quad (10)$$

$$\phi,\psi\ge 0.$$

Clearly, $\phi=0$ as $P>D$. If $P<P_{max}$ then $\psi=0$. The optimal values of P and Q can be obtained by solving the system of nonlinear equations (7) and (8). Nevertheless, we can't guarantee that the obtained solution will give the global minimum. If $P=P_{max}$ then the optimal value of Q can be obtained from (7). However, no feasible solution of the model exists if $P>P_{max}$.

4.1. Service level measurement

Let SL be the service level achieved when a safety stock S_f is maintained by the unreliable production system. By renewal reward theorem (Ross [2]) SL can be defined as

$$SL=\frac{E[\text{production amount per cycle}]}{D\cdot E[\text{cycle length}]}$$

The expected number of units produced per cycle is given by

$$\int_0^{Q/P}\left[\int_0^{(P-D)t/D}Pt\,g(s)\,ds+\int_{(P-D)t/D}^{\frac{S_f+(P-D)t}{D}}\frac{PDs}{P-D}g(s)\,ds\right.$$

$$\left.+\int_{\frac{S_f+(P-D)t}{D}}^{\infty}P\left(t+\frac{s_f}{P-D}\right)g(s)\,ds\right]dF(t)+\int_{Q/P}^{\infty}Q\,dF(t)$$

$$=A(P,Q)+B(P,Q)\,P\left(1-e^{-\frac{\mu S_f}{D}}\right) \quad \text{where}$$

$$A(P,Q)=\frac{P}{\lambda(P)}\left[1-\exp\{-\lambda(P)Q/P\}\right] \text{ and}$$

$$B(P,Q)=\frac{D\lambda(P)\left[1-\exp\left\{-\left(\lambda(P)+\frac{\mu(P-D)}{D}\right)\frac{Q}{P}\right\}\right]}{\mu(P-D)\left\{\lambda(P)+\frac{\mu(P-D)}{D}\right\}}.$$

$$E[\text{cycle length}]=\frac{P}{D\lambda(P)}\left[1-e^{\lambda(P)Q/P}\right]+$$

$$\frac{\lambda(P)\left\{P-De^{-\frac{\mu S_f}{D}}\right\}}{\mu(P-D)\left\{\lambda(P)+\frac{\mu(P-D)}{D}\right\}}\left[1-e^{-\left\{\lambda(P)+\frac{\mu(P-D)}{D}\right\}\frac{Q}{D}}\right].$$

Thus we have, $$SL=\frac{A(P,Q)+B(P,Q)\left[P-Pe^{-\frac{\mu S_f}{D}}\right]}{A(P,Q)+B(P,Q)\left[P-De^{-\frac{\mu S_f}{D}}\right]}$$

Note that $SL<1$ as $P>D$ and $SL\to 1$ as $S_f\to\infty$.

References

[1] M. Khouja and M. Mehrez, Economic production lot size model with variable production rate and imperfect quality, *Journal of the Operational Research Society*, 45 1405-1417, 1994.

[2] S. M. Ross, *Introduction to Probability Models*, 5th ed. New York, Academic Press, 1993.