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A Homogeneous Model for $\mathbb{P}_0$ and $\mathbb{P}_*$ Complementarity Problems

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1 Introduction

The homogeneous model for linear programs provides a most simple and firm theory in interior point algorithms. Andersen and Ye generalized this model to monotone complementarity problems (CPs) and showed that most of desirable properties can be inherited as long as the problem has the monotonicity[1]. Unfortunately, much dependence on the monotonicity prevents us from extending the model to more general problems, e.g., P_0 CPs or P_* CPs. In this article, we propose a new homogeneous model and its associated algorithm which have the following features: (a) The model preserves the P_0 (P_*) property if the original problem is a P_0 (P_*) CP. (b) The algorithm can be applied to P_0 CPs starting at a positive point near the central trajectory, and it does not need to use any big-$\mathcal{M}$ penalty parameter. (c) It generates a sequence that approaches feasibility and optimality simultaneously for any P_* CP having a complementarity solution, and (d) solves the P_* CP having a strictly feasible point.

2 Complementarity problems

The standard complementarity problem (CP) is given by

$$\text{(CP)}\quad \text{Find } (x,s)\in\Re^{2n}\quad \text{s.t. } s=f(x),\ (x,s)\geq 0,\ x_is_i=0\ (i\in N)$$

where f is a continuously differentiable function from $\Re^n_+ := \{x\in\Re^n:\ x\geq 0\}$ to $\Re^n$ and $N:=\{1,2,\ldots,n\}$.

A CP is said to be (asymptotically) feasible if and only if there is a bounded sequence $\{(x^k,s^k)\}\in\Re^{2n}_{++} := \{(x,s)\in\Re^{2n}:\ (x,s)>0\}$ such that $\lim_{k\to\infty} s^k - f(x^k)=0$, where any limit point $(\hat{x},\hat{s})$ of the sequence is called an (asymptotically) feasible point for the CP. In particular, an (asymptotically) feasible point $(\hat{x},\hat{s})$ satisfying $(\hat{x},\hat{s})>(0,0)$ is called an interior feasible point or a strictly feasible point.

Definition 2.1 *Let $\mathcal{K}$ be a subset of $\Re^n$ and $\kappa\geq 0$.*

(i) *$f:\mathcal{K}\to\Re^n$ is said to be a P_0 function if and only if for any $x^1\neq x^2\in\mathcal{K}$, there exists at least one index $i\in N$ such that*

$$(x_i^1-x_i^2)(f_i(x^1)-f_i(x^2))\geq 0.$$

(ii) *$f:\mathcal{K}\to\Re^n$ is said to be a $P_*(\kappa)$ function if and only if*

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+}(x_i^1-x_i^2)(f_i(x^1)-f_i(x^2)) + \sum_{i\in\mathcal{I}_-}(x_i^1-x_i^2)(f_i(x^1)-f_i(x^2))\geq 0$$

for any $x^1,x^2\in\mathcal{K}$, where $\mathcal{I}_+:=\mathcal{I}_+(x)$ and $\mathcal{I}_-:=\mathcal{I}_-(x)$ are a couple of index sets given by

$$\mathcal{I}_+(x):=\{i\in N:(x_i^1-x_i^2)(f_i(x^1)-f_i(x^2))\geq 0\},$$
$$\mathcal{I}_-(x):=\{i\in N:(x_i^1-x_i^2)(f_i(x^1)-f_i(x^2))< 0\}.$$

(iii) *$f:\mathcal{K}\to\Re^n$ is said to be a monotone function if and only if it is a $P_*(0)$ function.*

We say that the CP is monotone (respectively, $P_*(\kappa)$ or P_0) if f is monotone (respectively, $P_*(\kappa)$ or P_0). See the comprehensive books [2] for more details concerning CPs and algorithms.

3 A new homogeneous model

Andersen and Ye[1] provide the homogeneous *monotone* model (HMCP) related to (CP):

$$\text{(HMCP)}\quad \text{Find } (x,\tau,s,\kappa)\in\Re^{2(n+1)}$$
$$\text{s.t. } \begin{pmatrix} s\\ \kappa\end{pmatrix} = \phi(x,\tau) := \begin{pmatrix}\tau f(x/\tau)\\ -x^Tf(x/\tau)\end{pmatrix},$$
$$(x,\tau,s,\kappa)\geq 0,\ x_is_i=0\ (i\in N), \tau\kappa=0.$$

The map ϕ is monotone on the set $\Re^{n+1}_{++}$ if f is monotone on $\Re^n_+$. However, this fact does not necessarily hold for general cases. We introduce a new homogeneous model:

$$\text{(HCP)}\quad \text{Find } (x,t,s,u)\in\Re^{4n}$$
$$\text{s.t. } \begin{pmatrix} s\\ u\end{pmatrix} = \psi(x,t) := \begin{pmatrix}Tf(T^{-1}x)\\ -Xf(T^{-1}x)\end{pmatrix},$$
$$(x,t,s,u)\geq 0,\ x_is_i=0,\ t_iu_i=0\ (i\in N)$$

where $X:=\operatorname{diag} x_i (i\in N)$ and $T:=\operatorname{diag} t_i (i\in N)$. Let us define $z:=(x,t)\in\Re^{2n}$, $2N:=\{1,2,\ldots,2n\}$, and

$$\tau_C := \sup_{t^1,t^2\in C}\left\{\frac{\max\{t_i^1t_i^2:\ i\in N\}}{\min\{t_i^1t_i^2:\ i\in N\}}\right\}$$

for every nonempty subset C of $\Re^n_{++}$.

Lemma 3.1 (i) *If $f:\Re^n_+\to\Re^n$ is a P_0 function then $\psi:\Re^n_+\times\Re^n_{++}\to\Re^{2n}$ is a P_0 function.*

(ii) *Let C be a subset of $\Re^n_{++}$ with $0<\tau_C<\infty$. If f is a $P_*(\kappa)$ function for some $\kappa\geq 0$, then ψ is a $P_*(\kappa_C)$ function from $\Re^n_+\times C$ to $\Re^{2n}$ where κ_C satisfies $1+4\kappa_C=\tau_C(1+4\kappa)$.*

The following lemma gives a validity of solving (HCP) instead of (CP).

Lemma 3.2 **(i)** *(HCP) is (asymptotically) feasible and every (asymptotically) feasible solution is a complementarity solution.*

(ii) *Let (x^*, t^*, s^*, u^*) be a complementarity solution of (HCP). If $t^* > 0$, then $(T_*^{-1}x^*, T_*^{-1}s^*)$ is a complementarity solution for (CP).*

(iii) *Let $(\hat{x}, \hat{s})$ be a complementarity solution of (CP). Then, for every $t^* > 0$, we can construct an (asymptotically) feasible solution (x^*, t^*, s^*, u^*) i.e., a complementarity solution of (HCP) using $(\hat{x}, \hat{s})$.*

4 Main results

We summarize the main results obtained for (HCP).

Assumption 4.1 **(i)** *The original problem (CP) has a complementarity solution $(\hat{x}, \hat{s})$.*

(ii) *f is a $P_*(\kappa)$ function from $\Re^n_+$ to $\Re^n$.*

Theorem 4.2 *Suppose that Assumption 4.1 holds. Let $(\bar{a}, \bar{b}) \in \Re^{4n}_{++}$ and $\mathcal{T} := \{\theta(\bar{a}, \bar{b}) : \theta > 0\}$. Then*

$$\Psi^{-1}(\mathcal{T}) := \{(z, w) \in \Re^{4n}_{++} : \Psi(z, w) = \theta(\bar{a}, \bar{b}), \ \theta \in (0, 1]\}$$

forms a trajectory.

Assumption 4.3 *There exists an open subset C for which τ_C has a finite positive value and*

$$\Psi^{-1}(\mathcal{T}) \subset \Re^n_{++} \times C \times \Re^{2n}_{++}.$$

Theorem 4.4 *Suppose that Assumptions 4.1 and 4.3 are satisfied.*

(i) *The trajectory $\Psi^{-1}(\mathcal{T})$ is bounded.*

(ii) *Every limit point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of $\Psi^{-1}(\mathcal{T})$ is a complementarity solution of (HCP) with $t^* > 0$.*

Assumption 4.5 **(i)** *The original problem (CP) has a strictly feasible point $(\bar{x}, \bar{s})$.*

(ii) *f is a $P_*(\kappa)$ function from $\Re^n_+$ to $\Re^n$.*

Theorem 4.6 *Suppose that Assumption 4.5 holds. Then*

(i) *the trajectory $\Psi^{-1}(\mathcal{T})$ is bounded, and*

(ii) *every limit point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of $\Psi^{-1}(\mathcal{T})$ is a complementarity solution of (HCP) with $t^* > 0$.*

We provide an algorithm for tracing the trajectory. Let $z^0 = e$, $w^0 = 2\max_{i \in N}\{|\psi_i(z^0)|\}e$. Then, for every $\beta \in (0, 1)$, (z^0, w^0) satisfies $r^0 := w^0 - \psi(z^0) > 0$ and $\|Z^0 w^0 - \mu^0 e\| < \beta\mu^0$ where $\mu^0 := (z^0)^T w^0 / 2n$. At each iteration k with $(z^k, w^k) := (x^k, t^k, s^k, u^k)$, we set $r^k := w^k - \psi(z^k)$ and $\mu^k := (z^k)^T w^k / 2n$ and calculate a direction $(\Delta z^k, \Delta w^k)$ by solving

$$\Delta w^k - \nabla\psi(z^k)\Delta z^k = -\eta r^k,$$
$$Z_k \Delta w^k + W_k \Delta z^k = \gamma\mu^k e - Z_k w^k$$

where $\eta \in (0, 1)$ and $\gamma \in (0, 1)$. Define

$$g^k(\alpha) := \psi(z^k(\alpha)) - \psi(z^k) - \alpha\nabla\psi(z^k)\Delta z^k$$

and

$$z^k(\alpha) := z^k + \alpha\Delta z^k, \ w^k(\alpha) := w^k + \alpha\Delta w^k + g^k(\alpha).$$

By applying an inexact line search procedure, find an appropriate $\bar{\alpha} > 0$ and set $(z^{k+1}, w^{k+1}) := (z^k(\bar{\alpha}), w^k(\bar{\alpha}))$.

Theorem 4.7 *Suppose that Assumption 4.5 is satisfied.*

(i) *The algorithm is well defined and the generated sequence $\{(z^k, w^k)\}$ is bounded.*

(ii) *Every limit point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of $\{(z^k, w^k)\}$ is a solution of (HCP) with $t^* > 0$.*

(iv) *For every limit point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of $\{(z^k, w^k)\}$, $(T_*^{-1}x^*, T_*^{-1}s^*)$ is a complementarity solution of the original problem (CP).*

5 Concluding remarks

While numerous studies have been conducted on complementarity problems over cones in the last decade, there still remain several issues to be addressed: Extensions of the concepts of P_0 and P_* to the problems, developments of homogeneous algorithms for the problems, etc. A pioneer work can be seen in [3].

References

[1] E. Andersen and Y. Ye. On a homogeneous algorithm for the monotone complementarity problems. *Mathematical Programming*, 84:375–400, 1999.

[2] F. Facchinei and J.-S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems Volume I, II.* Springer-Verlag, 2003.

[3] M. S. Gowda, R. Sznajder and J. Tao. Some P-properties for linear transformations on Euclidean Jordan algebras. Technical Report TRGOW03-02, Department of Mathematics and Statistics, University of Maryland, 2003.