A multiple choice loss minimization problem with partial recall

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1 Introduction

To make the analysis simple, we consider our problem in the framework of the infinite formulation as defined and originally studied by Gianini and Samuels(1976). Let the best, second best, etc., of an infinite sequence of rankable applicants arrive at times $U_1, U_2, \cdots$, which are independent and uniformly distributed on the unit interval $(0,1]$. Let the interval $(0,1]$ be devided into n equal subintervals

$$I_k \equiv \left(\frac{k-1}{n}, \frac{k}{n}\right], \qquad k = 1, 2, \cdots, n.$$

Assume that the loss $q(i)$, possibly non-decreasing in i, is incurred if the i-th best applicant is employed. The D-M (decision maker) must choose a total of m applicants with the objective of minimizing the expected total loss incurred by the chosen applicants. Choice of the applicants can be made only at the end of each subinterval and the recall of the previous applicants, that appeared in that subinterval, is allowed. This choice may be based on the full memory of the relative ranks of all applicants observed so far.

Define the states of the process as follows:

(r,k) : state where there still remain r subintervals and the D-M must choose k more applicants in the future.

$(r,k;i_1,\cdots,i_k)$: state where the D-M is observing, at the end of I_{n-r+1}, that the ranks of best, 2nd best, $\cdots$, and k-th best in I_{n-r+1} relative to all their predecessors are $i_1, i_2, \cdots, i_k$ respectively $(i_1 < i_2 < \cdots < i_k)$.

Let v_k^r and $v_k^r(i_1,\cdots,i_k)$ denote the minimum expected losses starting from (r,k) and $(r,k;i_1,\cdots,i_k)$ respectively. To derive the optimality equation, we introduce the joint probability mass function and the loss function.

$p^r(i_1,\cdots,i_k)$: The joint probability mass function that the ranks of best, 2nd best, $\cdots$, k-th best in I_{n-r+1} relative to all their predecessors are $i_1, i_2, \cdots, i_k$ respectively, where $i_1 < i_2 < \cdots < i_k$.

$R_j(t)$: The expected loss incurred by choosing an applicant at time t whose rank relative to all its predecessors is j $(1 \le j,\ 0 < t \le 1)$.

These quantites are given as follows.

Lemma 1.
For $1 \le r < n$ and $i_1 < i_2 < \cdots < i_k$,

$$p^r(i_1,\cdots,i_k) = \left(\frac{n-r}{n-r+1}\right)^{i_k} \left(\frac{1}{n-r}\right)^k.$$

Lemma 2.
For $1 \le j$ and $0 < t \le 1$,

$$R_j(t) = \sum_{i=j}^{\infty} q(i) \binom{i-1}{j-1} t^j (1-t)^{i-j}.$$

Example 1: If the loss function is given by

$$q(i) = \begin{cases} 1, & \text{if } i = 1 \\ \dfrac{\beta(\beta+1)\cdots(\beta+i-2)}{(i-1)!}, & \text{if } i \ge 2, \end{cases}$$

for $\beta > 1$, then

$$R_j(t) = \frac{q(j)}{t^{\beta-1}}, \qquad j \ge 1.$$

It is noted that for $\beta = 2$, $q(i) \equiv i$, so that in this case the rank of the chosen applicant is regarded as loss.

Example 2: If the loss function is given by

$$q(i) = \begin{cases} 0, & \text{if } 1 \le i \le N \\ 1, & \text{if } i \ge N+1, \end{cases}$$

for some $N \ge 1$, then

$$R_j(t) = \begin{cases} 1 - \displaystyle\sum_{i=0}^{N-j} \binom{i+j-1}{i} t^j (1-t)^i, & \text{if} j \le N \\ 1, & \text{if} j > N, \end{cases}$$

Now, we have the following optimality equations

$$v_k^r(i_1,\cdots,i_k) = \min_{0\le j\le k}\left\{\sum_{t=1}^{j} R_{i_t}\left(\frac{n-r+1}{n}\right) + v_{k-j}^{r-1}\right\},$$

$$v_k^r = \sum\sum_{i_1<i_2<\cdots<i_k}\cdots\sum v_k^r(i_1,i_2,\cdots,i_k)$$
$$\times p^r(i_i,i_2,\cdots,i_k),\quad 1\le k\le m,\ 2\le r<n$$

with the boundary condition

$$v_k^1 = \sum_{j=1}^{k} R_j\left(\frac{1}{n}\right).$$

The expected total loss is given by

$$V_m^n = \min_{0\le j\le m}\left\{\sum_{i=1}^{j} R_i\left(\frac{1}{n}\right) + v_{m-j}^{n-1}\right\}.$$

Define

$$\phi_j^r(k) \equiv v_{k+1-j}^{r-1} - v_{k-j}^{r-1},\quad 1\le j\le k,$$

and also define, for $1\le j\le k$,

$$i_j^r(k) = \max\left\{i\ge j : R_i\left(\frac{n-r+1}{n}\right)\le \phi_j^r(k)\right\},$$

with $\max\{\phi\}=0$. We can now describe the form of the optimal policy as follows.

Theorem 3
There exists a sequence of decision numbers $\{i_j^r(k)\}_{j=1}^k$, such that the optimal decision in state $(r,k;i_1,\cdots,i_k)$ is to choose j-th best applicant provided $i_j\le i_j^r(k)$, irrespective of the values of $i_1,\cdots,i_{j-1},i_{j+1},\cdots,i_k, 1\le j\le k$. Moreover the sequence $\{i_j^r(k)\}_{j=1}^k$ satisfies the following monotonicity properties:
(i) $i_j^r(k)\ge i_{j+1}^r(k)$.
(ii) $i_j^r(k)\le i_j^r(k+1)$.
(iii) $i_j^r(k)\ge i_{j+1}^r(k+1)$. In particular, if $i_{j+1}^r(k+1)\ne 0$, then $i_j^r(k)=i_{j+1}^r(k+1)$.

2 Algorithm for calculating $\{i_j^r(k)\}$ and v_j^r

Let $p_j(i)$ be the probability that the rank of j-th best in I_{n-r+1} relative to all its predecessors is i. Then, for $i\ge j$

$$p_j(i) = \binom{i-1}{j-1}\left(\frac{1}{n-r+1}\right)^j\left(1-\frac{1}{n-r+1}\right)^{i-j}.$$

Algorithm

(i) Initialize $\quad v_i^1 = \sum_{j=1}^{i} R_j\left(\frac{1}{n}\right),\quad 1\le i\le m.$

(ii) Assume $\{v_i^{r-1}\}_{i=1}^m$ are given. Also assume that we are in state (r,k). First calculate

$$\varphi_i^r \equiv v_i^{r-1} - v_{i-1}^{r-1},\qquad 1\le i\le m,$$

and define, for fixed r and k,

$$\phi_j^r(k) = \varphi_{k+1-j}^r,\qquad 1\le j\le k,$$

and

$$K = \max\left\{j\ge 1 : R_j\left(\frac{n-r+1}{n}\right)\le \phi_j^r(k)\right\},$$

with $\max\{\phi\}=0$. Then the decision number is calculated as,

$$i_j^r(k) = \max\left\{i\ge j : R_i\left(\frac{n-r+1}{n}\right)\le \phi_j^r(k)\right\},$$

for $1\le j\le K$, and the probability that exactly the top j applicants are chosen is expressed as

$$q_j = \begin{cases} 1-\sum_{i=1}^{i_1^r(k)} p_1(i), & \text{if } j=0\\ \sum_{i=j}^{i_j^r(k)} p_j(i) - \sum_{i=j+1}^{i_{j+1}^r(k)} p_{j+1}(i), & \text{if } 1\le j<K\\ \sum_{i=K}^{i_K^r(k)} p_K(i), & \text{if } j=K.\end{cases}$$

Finally we obtain, for $K\ge 1$,

$$v_k^r = \sum_{j=1}^{K}\left(\sum_{i=j}^{i_j^r(k)} R_i\left(\frac{n-r+1}{n}\right)p_j^r(i)\right) + \sum_{j=0}^{K} v_{k-j}^{r-1} q_j.$$

When $K=0, v_k^r = v_k^{r-1}$.

References

[1] T. S. Ferguson: Who solved the secretary problem? *Statistical Science* 4. (1989) 282-289.

[2] J. Gianini and S. M. Samuels: The infinite secretary problem. *Annals of Probability* 4. (1976) 418-432.