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A revising method of unstable data in ANP by Bayes Theorem

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1. Introduction

We consider the simplest type of ANP that is composed of the set of criteria $C = \{C_1, \ldots, C_m\}$, the set of alternatives $A = \{A_1, \ldots, A_n\}$, the evaluation matrix U of alternatives by criteria and W of criteria by alternatives. This type of ANP has a so-called supermatrix

$$S = \begin{bmatrix} 0 & W \\ U & 0 \end{bmatrix}. \quad (1)$$

It is often said that the values of elements of W, the evaluation of criteria by alternatives, are unstable. Saaty [2] insists that Bayes Theorem is included in the framework of ANP. This study proposes the new approach of revising W based on this idea and to show no contradiction of the new one.

2. Structures of Bayes Theorem and ANP

We illustrate the structures of Bayes Theory and consider a group G of human beings (G may be the whole people of U.S.A.) Some of them have a cancer. Let C_1 be the set of persons of cancer and $C_2 = G \setminus C_1$ be the set of non-cancer ones. Denoting the percent/100 of $C_1(C_2)$ by $p_1(p_2)$, we have $p_1 + p_2 = 1$. Let A_1 be the set of persons who are decided to cancer by the medical checkup. And $A_2 = G \setminus A_1$ is the set of ones decided to have not cancer. Denoting the percent/100 of $A_1(A_2)$ by $q_1(q_2)$, then we have $q_1 + q_2 = 1$. Then we have the following four kinds of conditional probabilities:

$$u_{ji} = \frac{|A_j \cap C_i|}{|C_i|}, i, j = 1, 2. \quad (2)$$

All we can know is only the results of the medical check. The (conditional) probability for a person decided to have cancer by the medical check to have really cancer is clearly represented as

$$\frac{|A_1 \cap C_1|}{|A_1|}. \quad (3)$$

By using p_i, q_j and u_{ij}, then (3) is $\frac{u_{11}p_1}{q_1}$. In Bayes theory this ratio is called a posteriori probability. This way of expression is based on their idea taking C_1, C_2 as causes and A_1, A_2 as outcomes. The aposteriori probability w_{ij} of C_i on the outcome A_j is

$$w_{ij} = \frac{u_{ji}p_i}{q_j}. \quad (4)$$

Since $q_j = \sum_i u_{ji}p_i$, (4) is equivalent to $w_{ij} = \frac{u_{ji}p_i}{\sum_k u_{jk}p_k}$. This is the famous Bayes Theorem.

In order to have a linkage between Bayes Theorem and ANP, take the simplest actual example of (1) type of ANP. Consider two fast food companies A_1 and A_2, and two evaluation criteria C_1 and C_2.

Now assuming that the whole people G of U.S.A can be decomposed into two groups $\bar{C}_1$ supporting C_1 and $\bar{C}_2$ supporting C_2. The similar decomposition $\bar{A}_1$ and $\bar{A}_2$ is considered. Then evaluating weight $p_i(q_j)$ of $C_i(A_j)$ can be considered to be near percent/100 of $\bar{C}_i(\bar{A}_j)$ in G. Similarly evaluating weight u_{ji} of A_j by C_i can be considered to be near to the percent/100 of $\bar{A}_j$ within $\bar{C}_i$. Considering $p_i \approx |\bar{C}_i|/|G|$ and $q_i \approx |\bar{A}_j|/|G|$ and some realistic and mild assumptions, Saaty mentions that evaluating weight w_{ij} of C_i by A_j is close to $\frac{u_{ji}p_i}{q_j}$,

$$w_{ij} \approx \frac{u_{ji}p_i}{q_j}. \quad (5)$$

If (5) is valid with exact equality, it completely coincides with Bayes Theorem (4). This is a brief explanation of Saaty's claim "ANP includes Bayes Theorem".

3. The revising method of W

Our revising method, Bayes Revising Method(BRM), assumes the relations (5). To describe BRM, we define several symbols as follows:

$$U = \begin{bmatrix} u_{11} & \cdots & u_{1m} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nm} \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$W = \begin{bmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{m1} & \cdots & w_{mn} \end{bmatrix} = [w_1, \cdots, w_n]$$

W: initial value of evaluation matrix of criteria by alternatives($\boldsymbol{w}_j$ is an evaluating vector of criteria by A_j, $j = 1, \ldots, n$.)
$\boldsymbol{p} = [p_1, \ldots, p_m]^\top$: evaluation vector of criteria by an outer factor
$\boldsymbol{q} = [q_1, \ldots, q_n]^\top$: evaluation vector determined by $\boldsymbol{q} = U\boldsymbol{p}$.

Here we assume as usual ANP

$$\sum_{i=1}^{n} u_{ij} = 1, \quad \sum_{i=1}^{m} w_{ij} = 1, \quad w_{ij} \geq 0, u_{ij} \geq 0 \quad (6)$$

Writing (5) by matrix-forms, we have

$$W \approx (\Delta \boldsymbol{p}) U^\top (\Delta \boldsymbol{q})^{-1}, \quad (7)$$

where $\Delta \boldsymbol{p} = \begin{bmatrix} p_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & p_n \end{bmatrix}$. Considering $\boldsymbol{q} = U\boldsymbol{p}$, we can write the right hand-side of (7) as

$$\mathcal{W}[\boldsymbol{p}] = (\Delta \boldsymbol{p})\, U^\top\, (\Delta(U\boldsymbol{p}))^{-1} \quad (8)$$

which is considered to be a transformation of apriori probability into aposteriori probability $\mathcal{W}[\boldsymbol{p}]$ by Bayes Theorem. Here we call (8) Bayes transformation.

Now the principle of BRM is to make Bayes transformation $\mathcal{W}[\boldsymbol{p}]$ of the convex combination $\boldsymbol{p} = \sum_{j=1}^{n} r_j \boldsymbol{w}_j$ of $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$, to be nearest to W. That is, the principle of BRM is to find

$$\boldsymbol{p} = \sum_{j=1}^{n} r_j \boldsymbol{w}_j, \ \sum_{j=1}^{n} r_j = 1 \text{ and } r_j \geq 0 \quad (9)$$

such that

$$\mathcal{W}[\boldsymbol{p}] = (\Delta \boldsymbol{p}) U^\top (\Delta(U\boldsymbol{p}))^{-1} \quad (10)$$

is near to W as possible as we can. Then we take $\mathcal{W}[\boldsymbol{p}]$ as the revised W.

Here we take the min-max principle as the nearest; that is, the min-max principle is

$$\min \max_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \left\{ \frac{u_{ji} \sum_{k=1}^{n} w_{ik} r_k}{w_{ij} \sum_{k=1}^{n} (\boldsymbol{u}_j \boldsymbol{w}_k) r_k}, \frac{w_{ij} \sum_{k=1}^{n} (\boldsymbol{u}_j \boldsymbol{w}_k) r_k}{u_{ji} \sum_{k=1}^{n} w_{ik} r_k} \right\}$$

$$\text{s.t.} \sum_{k=1}^{n} r_k = 1, \ r_k \geq 0, \ k = 1, \ldots, n. \quad (11)$$

The optimization problem (11) is a typical fractional program and it can be solved by Dinkelbach algorithm [1].

Once we had the revised matrix $\hat{W}$, the analysis of ANP are carried out by the revised supermatrix

$$\hat{S} = \begin{bmatrix} 0 & \hat{W} \\ U & 0 \end{bmatrix}. \quad (12)$$

4. Some properties of BRM

Main properties of BRM are as follows:

Theorem 1 *The evaluation weight vector of criteria of ANP with the supermatrix (12) is an optimal solution* $\boldsymbol{p}^*$ *of (11) and the evaluation weight vector of alternatives is* $U\boldsymbol{p}^*$.

Theorem 2 *Let* S *be a supermatrix (1) and let* $\boldsymbol{p}^*$ *be an evaluation weight vector of criteria by applying BRM to* S. *Suppose that* W *of* S *satisfies* $w_{1j} > w_{2j} > \cdots > w_{mj}$ *for all* $j = 1, \ldots, n$, *then* $p_1^* > p_2^* > \cdots > p_m^*$. *That is, BRM has no contradiction.*

Theorem 3 *Let* λ^* *and* $\boldsymbol{p}^*$ *be the optimal value and the optimal solution of*

$$\min_{\boldsymbol{p} \in C(W)} \max_{\substack{i=1,\ldots,m, \\ j=1,\ldots,n}} \left\{ \frac{u_{ji} p_i}{w_{ij} \boldsymbol{u}_j \boldsymbol{p}}, \frac{w_{ij} \boldsymbol{u}_j \boldsymbol{p}}{u_{ji} p_i} \right\}, \quad (13)$$

respectively, where $C(W)$ *is the convex hull of* $\{\boldsymbol{w}_1, \cdots, \boldsymbol{w}_n\}$. *Let* $\bar{\lambda}$ *and* $\underline{\lambda}$ *be the optimal value of* $\min_{\boldsymbol{p}>0} \max_{\substack{i=1,\ldots,m, \\ j=1,\ldots,n}} \left\{ \frac{w_{ij} \boldsymbol{u}_j \boldsymbol{p}}{u_{ji} p_i} \right\}$ *and that of* $\max_{\boldsymbol{p}>0} \min_{\substack{i=1,\ldots,m, \\ j=1,\ldots,n}} \left\{ \frac{w_{ij} \boldsymbol{u}_j \boldsymbol{p}}{u_{ji} p_i} \right\}$, *respectively. Suppose that* $\lambda^* = \max\left\{\underline{\lambda}^{-1}, \bar{\lambda}\right\}$ *and that the optimal value* λ^* *of (13) and any optimal solution* $\boldsymbol{p}^*$ *of (13) satisfies* $\lambda^* > \max_{i=1,\ldots,m} \left\{ \frac{u_{ji} p_i^*}{w_{ij} \boldsymbol{u}_j \boldsymbol{p}^*}, \frac{w_{ij} \boldsymbol{u}_j \boldsymbol{p}^*}{u_{ji} p_i^*} \right\}$ *and* $\boldsymbol{p}^* \in C(\boldsymbol{w}_1, \cdots, \boldsymbol{w}_{j-1}, \boldsymbol{w}_{j+1}, \cdots, \boldsymbol{w}_n)$. *If a positive vector* $\tilde{\boldsymbol{w}}_j$ *satisfies* $\sum_{i=1}^{m} \tilde{w}_{ij} = 1$ *and* $\boldsymbol{p}^*$ *satisfies*

$$\lambda^* \geq \max_{i=1,\ldots,m} \left\{ \frac{\tilde{w}_{ij} \boldsymbol{u}_j \boldsymbol{p}^*}{u_{ji} p_i^*}, \frac{u_{ji} p_i^*}{\tilde{w}_{ij} \boldsymbol{u}_j \boldsymbol{p}^*} \right\},$$

then $\boldsymbol{p}^*$ *is also an optimal solution of*

$$\min_{\boldsymbol{p} \in C(\tilde{W})} \max \left\{ \begin{array}{l} \max\limits_{\substack{i=1,\ldots,m \\ l \neq j}} \left\{ \dfrac{u_{li} p_i}{w_{il} \boldsymbol{u}_l \boldsymbol{p}}, \dfrac{w_{il} \boldsymbol{u}_l \boldsymbol{p}}{u_{li} p_i} \right\} \\ \max\limits_{\substack{i=1,\ldots,m \\ l \neq j}} \left\{ \dfrac{u_{ji} p_i}{\tilde{w}_{ij} \boldsymbol{u}_j \boldsymbol{p}}, \dfrac{\tilde{w}_{ij} \boldsymbol{u}_j \boldsymbol{p}}{u_{ji} p_i} \right\} \end{array} \right\},$$

where $\tilde{W} = [\boldsymbol{w}_1, \cdots, \boldsymbol{w}_{j-1}, \tilde{\boldsymbol{w}}_j, \boldsymbol{w}_{j+1}, \cdots, \boldsymbol{w}_n]$.

References

[1] J. Borde and J. P. Crouzeix: Convergence of a Dinkelbach-Type Algorithm in Generalized Fractional Programming, *Zeitcshrift für Operations Research*, **31**(1987) 31-54.

[2] T.L. Saaty: *Analytic Network Process* (RWS, Pittsburgh, 2000).