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Alternative Randomization for Valuing American Options

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1 Introduction

Let $(S_t)_{t\geq 0}$ be the stock price governed by the risk-neutralized diffusion process

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dW_t, \quad t \geq 0$$

where $r > 0$ is the risk-free interest rate, $\delta \geq 0$ is a continuous dividend rate, $\sigma > 0$ is a volatility of the asset returns, and $(W_t)_{t\geq 0}$ is a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

We consider an American *put* option written on $(S_t)_{t\geq 0}$, which has maturity date T and strike price K. Let

$$P \equiv P(t, S_t) = P(t, S_t; K, r, \delta), \quad 0 \leq t \leq T$$

denote the value of the American put option at time t.

McKean [2] showed that the alive American put value P and an *early exercise boundary* (or a *critical stock price*) $(B_t)_{t\in[0,T]}$ can be jointly obtained by solving a *free boundary problem*, which is specified by the Black-Scholes-Merton PDE

$$\tfrac{1}{2}\sigma^2 S^2 P_{SS} + (r-\delta)SP_S - rP + P_t = 0, \quad S > B_t$$

together with the boundary conditions

$$\lim_{S\uparrow\infty} P(t,S) = 0$$
$$\lim_{S\downarrow B_t} P(t,S) = K - B_t$$
$$\lim_{S\downarrow B_t} P_S(t,S) = -1$$

and the terminal condition

$$P(T,S) = (K-S)^+.$$

2 Randomization Approach

Carr [1] developed a valuing method for the American put. Carr's randomization approach consists of the following steps:

1. Randomize the maturity date by an *exponentially* distributed random variable $\widetilde{T}$ with mean $E[\widetilde{T}] = \lambda^{-1} = T$ in order to value the so-called *Canadian option.*
2. Extend the result to the case that $\widetilde{T}$ is distributed as the *n-stage Erlangian* distribution with the same mean $E[\widetilde{T}] = T$.
3. Take the limit of the randomized option value by letting $n \to \infty$ to obtain the underlying American option value.

Actually, the idea of Carr's randomization is *not* new. In the theory of integral transforms, this idea goes by the name of the Post-Widder inversion formula: For a continuous function $g(t)$ $(t \geq 0)$, define

$$g_n^*(T) = \int_0^\infty g(t)\, \frac{(nt/T)^{n-1}}{(n-1)!}\frac{n}{T}\, e^{-nt/T} dt.$$

Then, we have

$$\lim_{n\to\infty} g_n^*(T) = g(T).$$

It is sometimes convenient to work with the equations where the current time t is replaced by the remaining time until maturity $s = T - t$. Let $\hat{P}(s, S_s) = P(T-s, S_{T-s})$ and for $\lambda > 0$ let

$$P^* \equiv P^*(\lambda, S) = \int_0^\infty \lambda e^{-\lambda s} \hat{P}(s,S) ds$$

be the Laplace-Carson transform (LCT) of $\hat{P}(s,S)$. Then, $P^*(\lambda, S)$ satisfies the ODE

$$\tfrac{1}{2}\sigma^2 S^2 P^*_{SS} + (r-\delta)SP^*_S - (\lambda+r)P^* + \lambda(K-S)^+ = 0, \quad S > L^*$$

together with the boundary conditions

$$\lim_{S\uparrow\infty} P^*(\lambda,S) = 0$$
$$\lim_{S\downarrow L^*} P^*(\lambda,S) = K - L^*$$
$$\lim_{S\downarrow L^*} P^*_S(\lambda,S) = -1.$$

The early exercise boundary $L^* \equiv L^*(\lambda)$ is given by the LCT of $\hat{B}_s = B_{T-s}$

$$L^*(\lambda) = \int_0^\infty \lambda e^{-\lambda s}\hat{B}_s ds,$$

which is a *constant* due to the memoryless property of the exponential distribution.

Theorem 1

$$P^*(\lambda,S) = \begin{cases} K - S, & S \leq L^* \\ \dfrac{\lambda}{\lambda+r}K - \dfrac{\lambda}{\lambda+\delta}S + c(S) + b(S) + d(S), & L^* < S < K \\ p(S) + b(S) + d(S), & S \geq K, \end{cases}$$

where

$$c(S) = \frac{1}{\theta_+ - \theta_-}\frac{\lambda}{\lambda+\delta}\left(1 - \frac{r-\delta}{\lambda+r}\theta_-\right)K\left(\frac{S}{K}\right)^{\theta_+}$$

$$p(S) = \frac{1}{\theta_+ - \theta_-}\frac{\lambda}{\lambda+\delta}\left(1 - \frac{r-\delta}{\lambda+r}\theta_+\right)K\left(\frac{S}{K}\right)^{\theta_-}$$

$$b(S) = -\frac{\theta_+}{\theta_-}c(L^*)\left(\frac{S}{L^*}\right)^{\theta_-}$$

$$d(S) = -\frac{1}{\theta_-}\frac{\delta}{\lambda+\delta}L^*\left(\frac{S}{L^*}\right)^{\theta_-}$$

and the parameters $\theta_\pm$ are roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2\theta^2 + (r - \delta - \tfrac{1}{2}\sigma^2)\theta - (\lambda + r) = 0.$$

Theorem 2
(i) The early exercise boundary L^* of the Canadian-American put option satisfies the equation

$$\lambda\left(\frac{L^*}{K}\right)^{\theta_+} = r(\theta_+ - 1) - \delta\theta_+\frac{L^*}{K}.$$

(ii) For the limiting case $\lambda \to 0$, we have

$$L^*(0) = \lim_{s\to\infty}\hat{B}_s = \frac{r(\theta_+^\circ - 1)}{\delta\theta_+^\circ}K = \frac{\theta_-^\circ}{\theta_-^\circ - 1}K,$$

where $\theta_\pm^\circ = \lim_{\lambda\to 0}\theta_\pm$. In addition, if $\delta = 0$, then

$$L^*(0) = \lim_{s\to\infty}\hat{B}_s = \frac{K}{1 + \frac{\sigma^2}{2r}}.$$

(iii) For the limiting case $\lambda \to \infty$, we have

$$\lim_{\lambda\to\infty}L^*(\lambda) = \hat{B}_0 = B_T = \min\left(\frac{r}{\delta}, 1\right)K.$$

3 New Randomization Based on Order Statistics

Let $X_1, \dots, X_{n+m}$ be *iid* random variables with an *exponential* distribution with parameter α (> 0), and let $X_{(i)}$ denote the i-th smallest of these random variables ($i = 1, \dots, n+m$). Then, the *pdf* of $X_{(n+1)}$ is

$$f(t) = \frac{(n+m)!}{n!(m-1)!}(1 - e^{-\alpha t})^n \alpha e^{-m\alpha t}, \quad t \geq 0.$$

If the modal value of $X_{(n+1)}$ is equal to T, *i.e.*,

$$M[X_{(n+1)}] \equiv \arg\max_t f(t) = \frac{1}{\alpha}\ln\frac{n+m}{m} = T,$$

then $X_{(n+1)}$ can be another candidate for the random maturity $\widetilde{T}$, because $\lim_{n,m\to\infty} V[X_{(n+1)}] = 0$.

For a continuous function $g(t)$ ($t \geq 0$) and $\alpha = \frac{1}{T}\ln\frac{n+m}{m}$, define

$$g^*_{n,m}(T) = \frac{(n+m)!}{n!(m-1)!}\int_0^\infty g(t)(1 - e^{-\alpha t})^n \alpha e^{-m\alpha t}dt.$$

Then, we have

$$\lim_{n,m\to\infty} g^*_{n,m}(T) = g(T).$$

Theorem 3 The sequence $(g^*_{n,m})_{n,m\geq 1}$ satisfies the recursion

$$g^*_{0,m}(T) = \textstyle\int_0^\infty m\alpha e^{-m\alpha t}g(t)dt$$

$$g^*_{n,m}(T) = \tfrac{n+m}{n}\, g^*_{n-1,m}(T) - \tfrac{m}{n}\, g^*_{n-1,m+1}(T), \quad n \geq 1.$$

For a set of the parameters $\{t, S, K, T, r, \delta, \sigma\}$, if we have a functional program for computing $P^*(\lambda, S)$ for any $\lambda \geq 0$, then the N-th randomized approximation $\pi_N \approx P(t, S)$ ($N \geq 1$) can be obtained by the following algorithm:

$\alpha = \frac{1}{T-t}\ln 2$
for $m = N$ **to** $2N$ **do**
  $g^*_{0,m} = P^*(m\alpha, S)$
next m
for $n = 1$ **to** N **do**
  for $m = N$ **to** $2N - n$ **do**
    $g^*_{n,m} = \frac{n+m}{n}g^*_{n-1,m} - \frac{m}{n}g^*_{n-1,m+1}$
  next m
next n
$\pi_N = g^*_{N,N}$

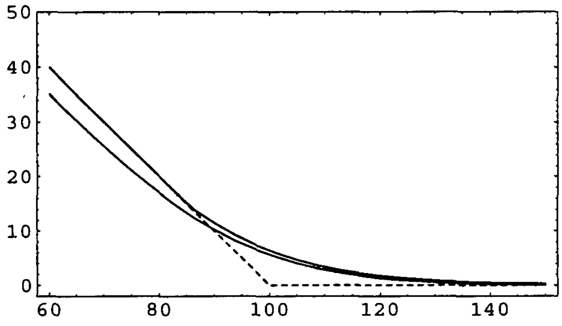

Figure 1: American & European put values
($t = 0, K = 100, T = 1, r = 0.05, \delta = 0, \sigma = 0.2$)

References

[1] Carr, P., Randomization and the American put, *Review of Financial Studies*, **11** (1998) 597–626.

[2] McKean, H.P., Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics, *Industrial Management Review*, **6** (1965) 32–39.