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# A Note on Expected Staying Time Related to Paging

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## 1 Introduction

The study of page replacement algorithms has a long history. It is still continued in other environments, such as web caching, distributed IP paging. Many replacement algorithms have been proposed over the years, including LRU, which evicts the least recently used page, FIFO, which evicts the earliest loaded page. Each performance has been evaluated on several analytical models, independent reference model (IRM), LRU stack model (LSM), or using competitive analysis. The models are too simple to reflect an actual program behavior, and some of them are suitable for a particular performance but others are not. For example, LRU and FIFO are equivalently evaluated in the competitive analysis[2] in contrast to our experience. Though their miss ratios can be explicitly derived from the IRM[1], it is difficult to prove the superiority of LRU. Thus we develop a hybrid model of IRM and LSM and show that an indirect performance measure, expected staying time, is useful on the model.

## 2 Expected Staying Time

The system under consideration has a two-level memory consisting of a *fast memory* of small size $m$, a large *slow memory*, and an input request *history*. Let $U = \{1, 2, \ldots, n\}$ be a set of all *pages* such that the page $i$ is referenced with probability $q_i$. A history $h = r_1, r_2, \ldots, r_t, \ldots$, where $r_t \in U$, is a reference string. If the requested page is in the fast memory, i.e., a *hit* occurs, then the reference incurs little cost. Otherwise, a *miss* occurs, then the reference is much more expensive.

The memory and the page can be modeled by a *stack* with $n$ *cells* $S = (s_1, s_2, \ldots, s_n)$ and an *item* $i \in U$ moving in $S$, respectively. We consider the random walk of an item $i$ starting from $s_1$ in the stack, where non-increasing reference probabilities are associated with the cells, called a *stack probability*. Notice that the item $i$ is referenced with probability $q_i$ (like IRM) and other items, whatever they are arranged, are referenced in accordance with the stack probability (like LSM). Thus our model is a hybrid model of IRM and LSM. An *expected staying time* $EST^X(i)$ of item $i$ is defined to be the expected time interval that the item $i$ stays in the fast memory $S_{fast} = (s_1, s_2, \ldots, s_m)$ under a paging scheme $X$. Let $e_j(i)$ be the expected time of item $i$ required for passing from the $j$-th cell to the $(j+1)$-st cell. Using it, we can represent

$$EST^X(i) = \sum_{j=1}^{m} e_j(i).$$

Let $M^X(t)$ be the contents of the fast memory at time $t$ when using a paging scheme $X$. In the long run, the evolution of $\{M^X(t)\}$ is described by a finite ergodic Markov chain. Thus there exist stationary probabilities for states $\{M^X\}$ and also $Pr\{i \in M^X\}$. Since the expected return time to the state $\{i \notin M^X\}$ is the inverse of $Pr\{i \notin M^X\}$, we have

$$Pr\{i \notin M^X\} = \frac{1}{1 + q_i \cdot EST^X(i)}.$$

We denote $E^X(i) = Pr\{i \notin M^X\}$ for simplicity. In our model, the reference probability $q_i$ of item $i$ is independent of whether it is contained in $S_{fast}$ or not, leading to the following definition.

**Definition 1** *A hit ratio $H^X$ for paging scheme $X$ is defined to be*

$$H^X = \sum_{i \in U} q_i \cdot Pr\{i \in M^X\}. \tag{1}$$

□

We can derive the following theorem.

**Theorem 1** *If*

$$\sum_{i \in U} q_i(E^X(i) - E^Y(i)) \geq 0 \tag{2}$$

*holds, we have $H^X \leq H^Y$.*

**Proof** Since we can calculate

$$\begin{aligned}
&\sum_{i \in U} q_i(E^X(i) - E^Y(i)) \\
&= \sum_{i \in U} q_i(\{1/(1 + q_i \cdot EST^X(i)) - 1/(1 + q_i \cdot EST^Y(i))\}) \\
&= \sum_{i \in U} q_i(Pr\{i \notin M^X\} - Pr\{i \notin M^Y\}) \\
&= \sum_{i \in U} q_i(Pr\{i \in M^Y\} - Pr\{i \in M^X\}),
\end{aligned}$$



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we obtain $H^X \leq H^Y$ from (1). □

Now we evaluate the schemes FIFO and LRU in terms of the expected staying time. Suppose that the reference probability of the $i$-th item is determined by a truncated geometric distribution $q_i = ap^{i-1}$, where $a = \frac{1-p}{1-p^n}$ and $0 < p < 1$. Let $c_j^X$ be the stack probability of the $j$-th cell $s_j$ for a paging scheme $X$, and $C_m^X$ the sum of the stack probability from $s_1$ through $s_m$ for $X$.

**Lemma 1** *For the scheme FIFO, the expected passing time $e_j(i)$ of the $i$-th item is*

$$e_j(i) = \frac{1}{1 + c_j^{FIFO} - C_m^{FIFO} - ap^{i-1}}.$$

□

**Lemma 2** *For the scheme LRU, the expected passing time $e_j(i)$ of the $i$-th item is*

$$e_j(i) = \frac{\prod_{l=2}^{j}(1 - C_{l-2}^{LRU})}{\prod_{l=1}^{j}(1 - ap^{i-1} - C_{l-1}^{LRU})},$$

*where we define $\prod_{l=2}^{1}(\text{any expression}) = 1$.* □

**Example 1** *Suppose that there are 3 items $x, y, z$, and that 2 items are in the fast memory. For example, $(x, y)$ means $x$ and $y$ are in the fast memory and $x$ is in the top cell. For FIFO, there are 3 states:*

$$(x, y), (z, x), (y, z) \quad or \quad (y, x), (z, y), (x, z),$$

*Let $p_x = a$, $p_y = ap$, $p_z = ap^2$ be access probabilities, which determines transition probabilities as depicted in Figure 1. For example, let $P_{xy}$ be the stationary probability of $(x, y)$.*

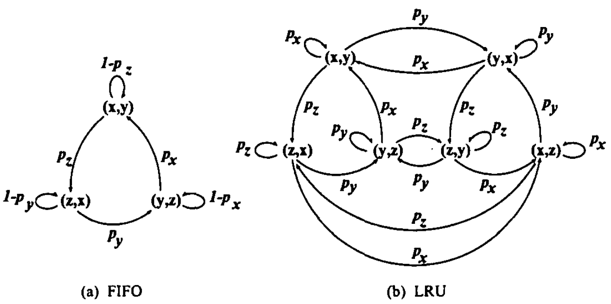


Figure 1: State transition diagrams

*Let $\pi^{FIFO} = (P_{xy}, P_{zx}, P_{yz})$, then*

$$\pi^{FIFO} = \pi^{FIFO} \begin{bmatrix} 1 - ap^2 & ap^2 & 0 \\ 0 & 1 - ap & ap \\ a & 0 & 1 - a \end{bmatrix}.$$

*Solving this equation with $P_{xy} + P_{zx} + P_{yz} = 1$, we have*

$$\pi^{FIFO} = (P_{xy}, P_{zx}, P_{yz}) = (a, ap, ap^2).$$

*The stack probabilities $c_1^{FIFO}$, $c_2^{FIFO}$ and $c_3^{FIFO}$ are*

$$\begin{aligned} c_1^{FIFO} &= P_{xy} \cdot p_x + P_{zx} \cdot p_z + P_{yz} \cdot p_y = a^2(1 + 2p^3) \\ c_2^{FIFO} &= P_{xy} \cdot p_y + P_{zx} \cdot p_x + P_{yz} \cdot p_z = a^2(2p + p^4) \\ c_3^{FIFO} &= P_{xy} \cdot p_z + P_{zx} \cdot p_y + P_{yz} \cdot p_x = 3a^2p^2 \end{aligned}$$

*We can similarly derive the stack probabilities for LRU. Using the expression (2), we obtain the following graph, showing that LRU outperforms FIFO.* □

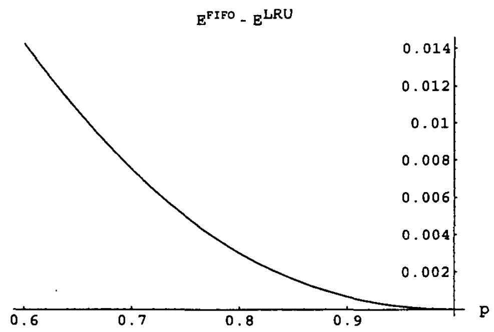


Figure 2: Comparison of LRU and FIFO

# 3 Conclusion

We proposed a new concept, an expected staying time, and investigated its relation to the hit ratio of paging schemes. The expected staying time is defined on a hybrid model, consisting of IRM and LSM. In addition, we verified its usefulness by applying to the comparison of FIFO and LRU.

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# References

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