Computational Aspects of an Extended EMQ Model with Variable Production Rate

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1. Introduction

Economic manufacturing quantity (EMQ) models for unreliable manufacturing systems have been developed in the literature even for general failure and general repair (corrective) time distributions [1,2]. However, in these studies, preventive maintenance (PM) has not been considered in a general way. Further, the efforts have been made to derive the production and maintenance policies for inflexible manufacturing systems, where the machine capacity is pre-determined. The purpose of this article is to formulate a generalized EMQ model for a flexible unreliable manufacturing system in which (i) the time to machine failure and repair (corrective and preventive) times follow general probability distributions and (ii) the machine failure rate is dependent on the production rate.

2. The EMQ Model

Nomenclature:

X: non-negative i.i.d. random variable denoting time to machine failure; $F_X(t)$: failure time distribution; $G_1(l_1)$: corrective repair time distribution; $G_2(l_2)$: preventive repair time distribution; p (>0): production rate (decision variable); $\overline{p}$ (>0): maximum production rate of the machine; d $(<p)$: demand rate; C_0 (>0): set up cost; C_1 (>0): corrective repair cost per unit time; C_2 $(<C_1)$: preventive repair cost per unit time; C_i (>0): holding cost per unit product per unit time; C_s (>0): shortage cost per unit product; Q (>0): order quantity (decision variable).

Model Formulation:

Consider a single-unit single-item production system in which at most one failure can occur in a production cycle. The process starts at time $t=0$ with the aim of producing a lot Q. If no failure occurs until time $t=Q/p$ then the process is stopped and PM is carried out. If, however, the machine fails before producing Q units then the corrective repair action is started immediately. During machine repair, the demand is met first from the accumulated inventory. If there is sufficient stock to meet the demand during machine repair then the next production starts when the on-hand inventory is exhausted. Shortages, if occur, are not met up after machine repair. To avoid an unrealistic decision making, we assume that $\underline{Q} \le Q \le \overline{Q}$, where $\underline{Q}$ and $\overline{Q}$ are lower and upper limits of Q, respectively. The mean time length of a cycle and the expected total cost for one cycle are given by

$$T(p,Q) = \int_0^\infty \mathrm{E}[\text{duration of a cycle} \mid X=t]\, dF_X(t)$$
$$= \int_0^{\frac{Q}{p}} \left[\int_0^{\frac{(p-d)t}{d}} \frac{pt}{d}\, dG_1(l_1) + \int_{\frac{(p-d)t}{d}}^\infty (t+l_1)\, dG_1(l_1) \right]$$
$$\times dF_X(t) + \int_{\frac{Q}{p}}^\infty \left[\int_0^{\frac{(p-d)Q}{pd}} \frac{Q}{d}\, dG_2(l_2) + \int_{\frac{(p-d)Q}{pd}}^\infty (Q/p$$
$$+ l_2)\, dG_2(l_2) \right] dF_X(t),$$

$$S(p,Q) = \int_0^\infty \mathrm{E}[\text{cost per cycle} \mid X=t]\, dF_X(t)$$
$$= C_0 + C_1 \int_0^{Q/p} \int_0^\infty l_1\, dG_1(l_1)\, dF_X(t)$$
$$+ C_2 \int_{Q/p}^\infty \int_0^\infty l_2\, dG_2(l_2)\, dF_X(t)$$
$$+ C_i \left[\int_0^{\frac{Q}{p}} \frac{(p-d)pt^2}{2d} dF_X(t) + \int_{\frac{Q}{p}}^\infty \frac{(p-d)Q^2}{2pd} dF_X(t) \right]$$
$$+ C_s d \int_0^{\frac{Q}{p}} \int_{\frac{(p-d)t}{d}}^\infty \left\{ l_1 - \frac{(p-d)t}{d} \right\} dG_1(l_1)\, dF_X(t)$$
$$+ C_s d \int_{\frac{Q}{p}}^\infty \int_{\frac{(p-d)Q}{pd}}^\infty \left\{ l_2 - \frac{(p-d)Q}{pd} \right\} dG_2(l_2) dF_X(t),$$

respectively. By the well-known renewal reward theorem, the expected cost per unit time in the steady state is given by

$$C(p,Q) = \lim_{t\to\infty} \frac{\mathrm{E}[\text{total cost on } (0,t]]}{t} = \frac{S(p,Q)}{T(p,Q)}. \quad (1)$$

The problem of our interest is

(P1) Minimize $C(p,Q)$, subject to
$h_1(p,Q) \equiv p-d>0$, $h_2(p,Q) \equiv \overline{p}-p \ge 0$,
$h_3(p,Q) \equiv Q-\underline{Q} \ge 0$, $h_4(p,Q) \equiv \overline{Q}-Q \ge 0$.

3. Development of Solution Algorithms:

If $\overline{p}, \underline{Q}$ and $\overline{Q}$ are specified in advance by the decision maker then a local optimal solution (p^*, Q^*) must satisfy the above inequality constraints and their corresponding multipliers m_1, m_2, m_3 and m_4 exist, satisfying the Kuhn-Tucker necessary conditions:

$$\left.\begin{aligned} \frac{\partial C(p,Q)}{\partial p} - m_1 + m_2 = 0 \\ \frac{\partial C(p,Q)}{\partial Q} - m_3 + m_4 = 0 \end{aligned}\right\} \quad (2)$$

$m_1(p-d)=0,\ \ m_2(\overline{p}-p)=0,\ \ m_3(Q-\underline{Q})=0,$ $m_4(\overline{Q}-Q)=0, p-d>0,\ \overline{p}-p\geq 0,\ Q-\underline{Q}\geq 0,$ $\overline{Q}-Q\geq 0,\ \ m_i\geq 0$ for $i=1,2,3,4$. Nevertheless, we can not guarantee analytically the existence of the global minimum, as the convex property of the objective function can not be proved. If (p^*,Q^*) is an interior point in the feasible region $D=\{d<p\leq\overline{p},\ \underline{Q}\leq Q\leq\overline{Q}\}$ then the generalized Newton's method can be applied to find it by solving the non-linear equations:

$$\left.\begin{aligned}\phi(p,Q)&\equiv\frac{\partial C(p,Q)}{\partial p}=0\\ \psi(p,Q)&\equiv\frac{\partial C(p,Q)}{\partial Q}=0\end{aligned}\right\}\qquad(3)$$

The associated Hessian matrix of equation (3) is

$$H=\begin{pmatrix}\frac{\partial\phi}{\partial p} & \frac{\partial\phi}{\partial Q}\\ \frac{\partial\psi}{\partial p} & \frac{\partial\psi}{\partial Q}\end{pmatrix}.$$

Defining $U(p,Q)=\psi(p,Q)\ (\partial\phi/\partial Q)-\phi(p,Q)\ (\partial\psi/\partial Q)$ and $V(p,Q)=\phi(p,Q)\ (\partial\psi/\partial p)-\psi(p,Q)\ (\partial\phi/\partial p)$, a computer algorithm for (p^*,Q^*) can be outlined as given below.

Algorithm 1:

Step 0. Input the model parameters and the accuracy parameter ϵ (>0).

Step 1. Set $(\hat{p},\hat{Q})$ as a candidate solution of simultaneous equations (3).

Step 2. Set $|\ H\ |$, the determinant of the Hessian matrix at $(\hat{p},\hat{Q})$.

Step 3. If $|\ H\ |=0$ then go to Step 7; otherwise, $h=U(\hat{p},\hat{Q})/\mid H\mid,\ \ k=V(\hat{p},\hat{Q})/\mid H\mid$.

Step 4. If h and k are both less than ϵ then go to Step 6; otherwise, go to Step 5.

Step 5. Set $\hat{p}=\hat{p}+h$, $\hat{Q}=\hat{Q}+k$ and go to Step 2.

Step 6. If $d<\hat{p}<\overline{p}$ and $\underline{Q}<\hat{Q}<\overline{Q}$ then assign $p^*=\hat{p}$ and $Q^*=\hat{Q}$. Otherwise, no interior point of D exists to minimize $C(p,Q)$. Stop.

Step 7. The method fails.

Alternatively, we can apply the log-barrier method (*e.g.*, see Bazaraa and Shetty [3]) to solve a sequence of unconstrained minimization problems of the form :

$$\text{Minimize } C_\mu(p,Q)=C(p,Q)-\mu\ B(p,Q)\qquad(4)$$

for a sequence of values of $\mu=\mu_k\downarrow 0$, where the barrier function is given by

$$B(p,Q)=\sum_{j=1}^{4}\ln\{h_j(p,Q)\}.$$

The limit as $\mu=\mu_k\downarrow 0$ of any convergent sequence $\{(p,Q):\mu>0\}$ is a local optimal solution of the problem (P1) and furthermore, $\mu B(p,Q)\downarrow 0$ as $\mu\downarrow 0^+$. The optimality conditions when minimizing $C_\mu(p,Q)$ are

$$\nabla C(p,Q)-\sum_{j=1}^{4}\frac{\mu}{h_j(p,Q)}\ \nabla h_j(p,Q)=\mathbf{0},\qquad(5)$$

where $\nabla=(\partial/\partial p,\partial/\partial Q)$ and $\mathbf{0}=(0,0)$. Based on the log-barrier method, we propose Algorithm 2.

Algorithm 2:

Step 0. Input the model parameters, the accuracy parameter ϵ (>0), the barrier parameter μ_0 (>0) and the reduction parameter θ $(0<\theta<1)$.

Step 1. Set $\mu\leftarrow\mu_0$ and $(\hat{p},\hat{Q})$ as a candidate solution of (4).

Step 2. Use Newton's method to calculate an approximation of the new target point $(\hat{p},\hat{Q})$ from (5).

Step 3. If $\mu<\epsilon$ then stop after assigning $p^*=\hat{p}$ and $Q^*=\hat{Q}$. Otherwise, set $\mu\leftarrow\theta\ \mu$; goto Step 2.

If (p^*,Q^*) is on the boundary of D, then the problem reduces to an one dimensional optimization problem which can be handled easily.

4. Numerical Example

We consider the exponential failure distribution [4]: $F_X(t)=1-\exp\{-\lambda(p)t\},\ \ t>0$, in which the failure rate $\lambda(p)=\alpha p^\beta$; α,β being real positive constants. $\lambda(p)$ is a concave (convex) function of p, for $\beta<1$ $(\beta>1)$. Since the maximum production rate $(\overline{p})$ of the machine is pre-assumed, we calculate the optimal policy for only $0<\beta<1$. Let the parameter values be: $d=50$, $C_i=0.5$, $C_s=1.25$, $C_0=500$, $C_1=250$, $C_2=50$, $\mu_1=4$, $\mu_2=10$, $\alpha=0.3$, $\beta=0.005$, $\theta=10^{-1}$, $\epsilon=10^{-6}$, $\underline{Q}=300$, $\overline{Q}=900$ and $\overline{p}=300$. Using Algorithm 1, a local optimal solution is obtained as $p^*=85.19$ and $Q^*=693.06$ which are also found as the target values in Algorithm 2 (log-barrier method). See Table 1.

Table 1 Convergence of the solution sequence in log-barrier method.

μ	$\hat{p}$	$\hat{Q}$	$C_\mu(\hat{p},\hat{Q})$	$\mu B(\hat{p},\hat{Q})$
10^2	167.29	579.05	-1945.68	2105.5623
10^1	128.93	536.85	-53.57	208.7288
10^0	92.56	622.13	133.11	20.4879
10^{-1}	86.20	679.61	151.47	2.0289
10^{-2}	85.30	691.50	153.30	0.2024
10^{-3}	85.20	692.87	153.48	0.0202
10^{-4}	85.19	693.01	153.50	0.0020
10^{-5}	85.19	693.02	153.50	0.0002

References

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