A Hybrid Measure of Efficiency in DEA *

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1. Introduction

There are two types of measures or approaches in DEA: *radial* and *non-radial*. Difference exits in the characterization of input or output items. Suppose that there are four inputs x_1, x_2, x_3 and x_4 in the concerned problem, where x_1 and x_2 are radial, and x_3 and x_4 are non-radial. That is, (x_1, x_2) are subject to change proportionally, such as $(\alpha x_1, \alpha x_2)$ (with $\alpha > 0$), while x_3 and x_4 are subject to change non-radially. These differences should be reflected in the evaluation of efficiency. The radial input part (x_1, x_2) satisfies the efficiency status if there is no proportionally reduced input $(\alpha x_1, \alpha x_2)$ (with $\alpha < 1$) that can produce the observed outputs. The non-radial input part x_3 (x_4) satisfies the efficiency status if there is no reduced x_3 (x_4) that can produce the observed outputs. Analogously, the output part can be divided into the radial and non-radial outputs.

The radial approach is represented by the CCR (Charnes, Cooper and Rhodes (1978)) and BCC (Banker, Charnes and Cooper (1984)) models. Its shortcoming is that it neglects the non-radial input/output slacks. The non-radial approach includes Russell (1985), Pastor et al. (1999) and Tone (2001). It deals with slacks directly but it neglects the radial characteristics of inputs and/or outputs.

In this paper, we integrate them in a unified framework and propose a hybrid measure of efficiency.

2. A hybrid measure

Let the observed input and output data matrices be $X \in R_+^{m\times n}$ and $Y \in R_+^{s\times n}$, respectively, where n, m and s designate the numbers of DMUs (decision making units), inputs and outputs. We decompose the input matrix into the radial part $X^R \in R_+^{m_1\times n}$ and non-radial part $X^{NR} \in R_+^{m_2\times n}$ with $m = m_1 + m_2$, as follows:

$$X = \begin{pmatrix} X^R \\ X^{NR} \end{pmatrix}. \tag{1}$$

Analogously, we decompose the output matrix Y into the radial part $Y^R \in R^{s_1\times n}$ and the non-radial part $Y^{NR} \in R^{s_2\times n}$ with $s = s_1 + s_2$, as follows:

$$Y = \begin{pmatrix} Y^R \\ Y^{NR} \end{pmatrix}. \tag{2}$$

We assume that the data set is positive, i.e., $X > 0$ and $Y > 0$. The production possibility set P is defined by

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq X\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}, \tag{3}$$

where $\boldsymbol{\lambda}$ is a nonnegative vector in R^n. (We can impose some constraints to $\boldsymbol{\lambda}$, such as $\sum_{j=1}^n \lambda_j = 1$ (the variable returns to scale model).)

We consider an expression for describing a certain DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o) = (\boldsymbol{x}_o^R, \boldsymbol{x}_o^{NR}, \boldsymbol{y}_o^R, \boldsymbol{y}_o^{NR}) \in P$ as

$$\begin{aligned} \theta \boldsymbol{x}_o^R &= X^R\boldsymbol{\lambda} + \boldsymbol{s}^{R-} \\ \boldsymbol{x}_o^{NR} &= X^{NR}\boldsymbol{\lambda} + \boldsymbol{s}^{NR-} \\ \phi \boldsymbol{y}_o^R &= Y^R\boldsymbol{\lambda} - \boldsymbol{s}^{R+} \\ \boldsymbol{y}_o^{NR} &= Y^{NR}\boldsymbol{\lambda} - \boldsymbol{s}^{NR+}, \end{aligned} \tag{4}$$

with $\theta \leq 1$, $\phi \geq 1$, $\boldsymbol{\lambda} \geq \mathbf{0}$, $\boldsymbol{s}^{R-} \geq \mathbf{0}$, $\boldsymbol{s}^{NR-} \geq \mathbf{0}$, $\boldsymbol{s}^{R+} \geq \mathbf{0}$, $\boldsymbol{s}^{NR+} \geq \mathbf{0}$. The vectors $\boldsymbol{s}^{R-} \in R^{m_1}$ and $\boldsymbol{s}^{NR-} \in R^{m_2}$ indicate the *excesses* for the radial and non-radial inputs, respectively, while $\boldsymbol{s}^{R+} \in R^{s_1}$ and $\boldsymbol{s}^{NR+} \in R^{s_2}$ indicate the *shortfalls* for the radial and non-radial outputs, respectively. They are called *slacks*.

Apparently, $\theta = 1,\ \phi = 1, \lambda_o = 1, \lambda_j = 0 (\forall j \neq o)$ with all slacks being zero is a feasible expression. Based on the expression (4), we define an index ρ as follows:

$$\rho = \frac{1 - \frac{m_1}{m}(1-\theta) - \frac{1}{m}\sum_{i=1}^{m_2} s_i^{NR-}/x_{io}^{NR}}{1 + \frac{s_1}{s}(\phi - 1) + \frac{1}{s}\sum_{r=1}^{s_2} s_r^{NR+}/y_{ro}^{NR}}. \tag{5}$$

This index ρ is designed so that it is decreasing with respect to decrease in θ and increase in ϕ, s_i^{NR-} $(\forall i)$ and s_r^{NR+} $(\forall r)$, but is not affected by $\boldsymbol{s}^{R-}$ and $\boldsymbol{s}^{R+}$ directly, reflecting free disposability of these radial slacks. This index is also units invariant, i.e., invariant with respect to the measurement units of data.

The hybrid efficiency status of the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o) = (\boldsymbol{x}_o^R, \boldsymbol{x}_o^{NR}, \boldsymbol{y}_o^R, \boldsymbol{y}_o^{NR}) \in P$ is defined as follows:

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Definition 1 (Hybrid efficient status) *The DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is hybrid efficient if and only if $\rho = 1$ for every feasible expression of (4), i.e., $\theta = 1,\ \phi = 1,\ \boldsymbol{s}^{NR-} = \boldsymbol{0},\ \boldsymbol{s}^{NR+} = \boldsymbol{0}$.*

This status can be identified by solving the following program with the variables $\theta,\ \phi,\ \boldsymbol{\lambda},\ \boldsymbol{s}^{NR-},\ \boldsymbol{s}^{NR+}$.

$$
\begin{aligned}
&[\text{Hybrid}]\\
&\rho^* = \min \rho \text{ in } (5) \qquad (6)\\
&\text{st. } \theta \boldsymbol{x}_o^R \geq X^R \boldsymbol{\lambda}\\
&\boldsymbol{x}_o^{NR} = X^{NR}\boldsymbol{\lambda} + \boldsymbol{s}^{NR-}\\
&\phi \boldsymbol{y}_o^R \leq Y^R \boldsymbol{\lambda}\\
&\boldsymbol{y}_o^{NR} = Y^{NR}\boldsymbol{\lambda} - \boldsymbol{s}^{NR+}\\
&\theta \leq 1,\ \phi \geq 1,\ \boldsymbol{\lambda} \geq \boldsymbol{0},\ \boldsymbol{s}^{NR-} \geq \boldsymbol{0},\ \boldsymbol{s}^{NR+} \geq \boldsymbol{0}.
\end{aligned}
$$

Let an optimal solution for this program be $(\theta^*, \phi^*,\ \boldsymbol{\lambda}^*,\ \boldsymbol{s}^{NR-*},\ \boldsymbol{s}^{NR+*})$. Then we have a theorem:

Theorem 1 *The DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is hybrid-efficient if and only if $\rho^* = 1$, i.e., $\theta^* = 1, \phi^* = 1,\ \boldsymbol{s}^{NR-*} = \boldsymbol{0},\ \boldsymbol{s}^{NR+*} = \boldsymbol{0}$.*

The [Hybrid] can be transformed into a linear program using the Charnes-Cooper transformation (Charnes and Cooper (1962)).

For a hybrid-inefficient DMU, i.e., $\rho^* < 1$, the hybrid-*projection* is given by

$$\bar{\boldsymbol{x}}_o^R \leftarrow \theta^* \boldsymbol{x}_o^R \qquad (7)$$

$$\bar{\boldsymbol{x}}_o^{NR} \leftarrow \boldsymbol{x}_o^{NR} - \boldsymbol{s}^{NR-*} \qquad (8)$$

$$\bar{\boldsymbol{y}}_o^R \leftarrow \phi^* \boldsymbol{y}_o^R \qquad (9)$$

$$\bar{\boldsymbol{y}}_o^{NR} \leftarrow \boldsymbol{y}_o^{NR} + \boldsymbol{s}^{NR+*}. \qquad (10)$$

We notice that the radial slacks $\boldsymbol{s}^{R-*}$ and $\boldsymbol{s}^{R+*}$, if they exist, are not accounted in the above projection, since they are assumed to be free disposable and have no effect in efficiency evaluation. Using the optimal solution $(\theta^*, \phi^*, \boldsymbol{s}^{NR-*}, \boldsymbol{s}^{NR+*})$, we can decompose the hybrid efficiency indicator ρ^* into four factors as follows:

Radial input ineff.: $\alpha_1 = \dfrac{m_1}{m}(1 - \theta^*)$

Non-radial input ineff.: $\alpha_2 = \dfrac{1}{m}\displaystyle\sum_{i=1}^{m_2} s_i^{NR-*}/x_{io}^{NR}$

Radial output ineff.: $\beta_1 = \dfrac{s_1}{s}(\phi^* - 1)$

Non-radial output ineff.: $\beta_2 = \dfrac{1}{s}\displaystyle\sum_{r=1}^{s_2} s_r^{NR+*}/y_{ro}^{NR}$.

Also we define input and output inefficiencies as:

Input inefficiency: $\alpha = \alpha_1 + \alpha_2 \qquad (11)$

Output inefficiency: $\beta = \beta_1 + \beta_2. \qquad (12)$

Thus, ρ^* can be expressed as:

$$\rho^* = \frac{1-\alpha}{1+\beta} = \frac{1-\alpha_1-\alpha_2}{1+\beta_1+\beta_2}. \qquad (13)$$

This expression is useful for finding the sources of inefficiency and the magnitude of their influence on the efficiency score ρ^*.

3. Economic interpretations

The dual program corresponding to the linear program [LP] can be described in terms of dual variables $\boldsymbol{v}^R \in R^{m_1},\ \boldsymbol{v}^{NR} \in R^{m_2},\ \boldsymbol{u}^R \in R^{s_1},\ \boldsymbol{u}^{NR} \in R^{s_2},\ w \in R$ as follows:

$$
\begin{aligned}
&[\text{Dual}]\ w^* = \max\ w \qquad (14)\\
&\text{st. } w = 1 - \boldsymbol{v}^R\boldsymbol{x}_o^R - \boldsymbol{v}^{NR}\boldsymbol{x}_o^{NR}\\
&+\boldsymbol{u}^R\boldsymbol{y}_o^R + \boldsymbol{u}^{NR}\boldsymbol{u}_o^{NR}\\
&-\boldsymbol{v}^R X^R - \boldsymbol{v}^{NR}X^{NR}\\
&+\boldsymbol{u}^R Y^R + \boldsymbol{u}^{NR}Y^{NR} \leq \boldsymbol{0} \qquad (15)\\
&\boldsymbol{v}^{NR} \geq \frac{1}{m}[1/\boldsymbol{x}_o^{NR}] \qquad (16)\\
&\boldsymbol{u}^{NR} \geq \frac{w}{s}[1/\boldsymbol{y}_o^{NR}] \qquad (17)\\
&\boldsymbol{v}^R\boldsymbol{x}_o^R \geq \frac{m_1}{m} \qquad (18)\\
&\boldsymbol{u}^R\boldsymbol{y}_o^R \geq \frac{s_1}{s}w \qquad (19)\\
&\boldsymbol{v}^R \geq \boldsymbol{0},\ \boldsymbol{u}^R \geq \boldsymbol{0}, \qquad (20)
\end{aligned}
$$

where the notation $[1/\boldsymbol{x}_o^{NR}]$ designates the row vector $(1/x_{1o}^{NR}, \cdots, 1/x_{m_2 o}^{NR})$.

The dual variables $\boldsymbol{v}^R \in R^{m_1},\ \boldsymbol{v}^{NR} \in R^{m_2},\ \boldsymbol{u}^R \in R^{s_1}$, and $\boldsymbol{u}^{NR} \in R^{s_2}$ can be interpreted as the virtual unit-costs and unit-prices of the corresponding input and output items, respectively. The dual program aims at finding the optimal virtual unit-costs and unit-prices for the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ so that the profit $\boldsymbol{u}^R\boldsymbol{y}_j^R + \boldsymbol{u}^{NR}\boldsymbol{y}_j^{NR} - \boldsymbol{v}^R\boldsymbol{x}_j^R - \boldsymbol{v}^{NR}\boldsymbol{x}_j^{NR}$ does not exceed zero for any DMU (including $(\boldsymbol{x}_o, \boldsymbol{y}_o)$), and maximizes the profit $\boldsymbol{u}^R\boldsymbol{y}_o^R + \boldsymbol{u}^{NR}\boldsymbol{y}_o^{NR} - \boldsymbol{v}^R\boldsymbol{x}_o^R - \boldsymbol{v}^{NR}\boldsymbol{x}_o^{NR}$ for the DMU_o concerned. Apparently, the optimal profit is at best zero and hence $w^* = 1$ for the hybrid efficient DMU.

Constraints (16) and (17) restrict the feasible virtual unit-cost and virtual unit-price $\boldsymbol{v}^{NR}$ and $\boldsymbol{u}^{NR}$ of the non-radial inputs and outputs to the positive orthant, respectively, while constraints (18) and (19) set the lower bound to the radial cost $\boldsymbol{v}^R\boldsymbol{x}_o^R$ and the radial price $\boldsymbol{u}^R\boldsymbol{y}_o^R$, respectively.

Reference

Charnes and Cooper (1962) NRLQ 15, 333-334.
Charnes et al. (1978) EJOR 2, 429-444.
Banker et al. (1984) Mgmt Sci 30, 1078-1092.
Tone (2001) EJOR 130, 498-509.