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# A Homogeneous Model for Mixed Complementarity Problems over Symmetric Cones

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## 1 Introduction

In this article, we propose a homogeneous model for mixed complementarity problems (MCPs) over symmetric cones, and discuss its theoretical aspects. The model is a natural extension of the model in [1] for monotone complementarity problems over the nonnegative orthant in $\Re^n$. The analyses are based on [4] where the weighted paths for nonlinear monotone MCPs are studied. We show the existence of a path having the following properties: (a) The path is bounded and has a trivial starting point without any regularity assumption concerning the existence of feasible or strictly feasible solutions. (b) Any accumulation point of the path is a solution of the homogeneous model. (c) If the original problem is solvable, then every accumulation point of the path gives us a finite solution. (d) If the original problem is strongly infeasible, then every accumulation point of the path gives us a finite certificate proving infeasibility.

## 2 MCPs over symmetric cones

Let $V$ be an $n$-dimensional real vector space and $(V,\gamma)$ be a Euclidian Jordan algebra with an identity element $e$. We denote by $K$ a symmetric cone of $V$ which is a self-dual closed convex cone. It is known that a cone in $V$ is symmetric if and only if it is the cone of squares of $V$ given by $K=\{x\circ x:\ x\in V\}$ (cf. [2]).

The MCP over the symmetric cone is given by

$$\begin{aligned}&\text{Find}\quad (x,y,z)\in K\times K\times\Re^m,\\ &\text{s.t.}\quad F(x,y,z)=0,\ x\circ y=0.\end{aligned}\qquad(1)$$

where $F$ is a continuously differentiable function from $K\times K\times\Re^m$ to $V\times\Re^m$.

Observe the following nonlinear semidefinite program:

$$\text{Min}\ \ \theta(x)\ \text{s.t.}\ \ G(x)\in -S^n_+,\ \ h(x)=0 \qquad (2)$$

where $S^n_+$ is the cone of semidefinite matrices in the set $S^n$ of $n\times n$ symmetric matrices, $\theta:\Re^m\to\Re$, $G:\Re^m\to S^n$ and $h:\Re^m\to\Re^p$ are given smooth mappings. It has been shown that an optimality condition for (2) can be formulated as a problem of the form (1) under an appropriate constraint qualification[3].

We say that an MCP is

- *(asymptotically) feasible* iff there exists a bounded sequence $\{x^{(k)},y^{(k)},z^{(k)}\}\subseteq \mathrm{int}K\times\mathrm{int}K\times\Re^m$ such that $\lim_{k\to\infty}F(x^{(k)},y^{(k)},z^{(k)})=0$,
- *strongly infeasible* iff $0\notin \mathrm{cl}(F(K\times K\times\Re^m))$.

We use the following assumption on the map $F$.

**Assumption 2.1 (i)** *$F$ is $(x,y)$-equilevel-monotone on its domain, i.e., for every $(x,y,z)$ and $(x',y',z')$ in the domain of $F$ satisfying $F(x,y,z)=F(x',y',z')$, $\langle x-x',y-y'\rangle\ge 0$ holds.*

**(ii)** *$F$ is $z$-bounded on its domain, i.e., for every $\{(x^{(k)},y^{(k)},z^{(k)})\}$ in the domain of $F$, if $\{(x^{(k)},y^{(k)})\}$ and $\{F(x^{(k)},y^{(k)},z^{(k)})\}$ are bounded then the sequence $\{z^{(k)}\}$ is also bounded.*

**(iii)** *$F(x,y,z)$ is $z$-injective on its domain, i.e., if $(x,y,z)$ and $(x,y,z')$ lie in the domain of $F$ and satisfy $F(x,y,z)=F(x,y,z')$ then $z=z'$.*

## 3 A homogeneous model

In this section, we give a homogenous model for solving MCPs where the map $F:K\times K\times\Re^m\to V\times\Re^m$ is of the form

$$F(x,y,z)=(y-\psi_1(x,z),\psi_2(x,z))$$

and $\psi:=(\psi_1,\psi_2):K\times\Re^m\to V\times\Re^m$. For the problem, we consider the following homogeneous model(HMCP):

$$\begin{aligned}&\text{Find}\quad (x,\tau,y,\kappa,z)\in(K\times\Re_{++})\times(K\times\Re_+)\times\Re^m,\\ &\text{s.t.}\quad F_{\mathrm H}(x,\tau,y,\kappa,z)=0,(x,\tau)\circ_{\mathrm H}(y,\kappa)=0\end{aligned}$$

where $F_{\mathrm H}:(K\times\Re_{++})\times(K\times\Re_+)\times\Re^m\to(V\times\Re)\times\Re^m$ and $(x,\tau)\circ_{\mathrm H}(y,\kappa)$ are given by

$$F_{\mathrm H}(x,\tau,y,\kappa,z):=\begin{pmatrix} y-\tau\psi_1(x/\tau,z/\tau)\\ \kappa+\langle\psi_1(x/\tau,z/\tau),x\rangle+\psi_2(x/\tau,z/\tau)^Tz\\ \tau\psi_2(x/\tau,z/\tau)\end{pmatrix}$$

and

$$(x,\tau)\circ(y,\kappa) := \begin{pmatrix} x\circ y \\ \tau\kappa \end{pmatrix}.$$

We also define the scalar product $\langle (x,\tau),(y,\kappa)\rangle_{\rm H}$ by

$$\langle (x,\tau),(y,\kappa)\rangle_{\rm H} := \langle x,y\rangle + \tau\kappa.$$

For ease of notation, we use the following symbols: $V_{\rm H} := V\times\Re$, $K_{\rm H} := K\times\Re_{+}$, $x_{\rm H} := (x,\tau)\in V_{\rm H}$, $y_{\rm H} := (y,\kappa)\in V_{\rm H}$, and define $\psi_{\rm H} := (\psi_{\rm H1},\psi_{\rm H2})$ by

$$\psi_{\rm H1}(x_{\rm H},z) := \begin{pmatrix} \tau\psi_1(x/\tau,z/\tau) \\ -\langle\psi_1(x/\tau,z/\tau),x\rangle - \psi_2(x/\tau,z/\tau)^T z \end{pmatrix},$$

$$\psi_{\rm H2}(x_{\rm H},z) := \tau\psi_2(x/\tau,z/\tau).$$

We can easily see that $\mathrm{int}K_{\rm H} = \mathrm{int}K_{\rm H}\times\Re_{++}$ and

$$F_{\rm H}(x_{\rm H},y_{\rm H},z) = \begin{pmatrix} y_{\rm H} - \psi_{\rm H1}(x_{\rm H},z) \\ \psi_{\rm H2}(x_{\rm H},z) \end{pmatrix}.$$

Note that since $K_{\rm H} = \{x_{\rm H}^2 = (x^2,\tau^2) : x_{\rm H}\in V_{\rm H}\}$, the closed convex cone $K_{\rm H}$ is the symmetric cone of $V_{\rm H}$.

# 4 Main results

The following theorems are the main results.

**Theorem 4.1** *Suppose that $\psi_{\rm H}$ satisfies Assumption 2.1.*

**(i)** *The HMCP is asymptotically feasible and solvable.*

**(ii)** *The MCP has a solution if and only if the HMCP has an asymptotical solution $(x_{\rm H}^*, y_{\rm H}^*, z^*) = (x^*,\tau^*,y^*,\kappa^*,z^*)$ with $\tau^* > 0$. In this case, $(x^*/\tau^*, y^*/\tau^*, z^*/\tau^*)$ is a solution of (CP).*

**(iii)** *The MCP is strongly infeasible if and only if the HMCP has an asymptotical solution $(x^*,\tau^*,y^*,\kappa^*,z^*)$ with $\kappa^* > 0$.*

**Theorem 4.2** *Suppose that $\psi_{\rm H}$ satisfies Assumption 2.1.*

**(i)** *The set*

$$P := \{(x_{\rm H}(t), y_{\rm H}(t), z(t)) : x_{\rm H}\circ_{\rm H} y_{\rm H} = te, F(x_{\rm H},y_{\rm H},z) = tF(e,e,0),\quad t\in(0,1]\}$$

*forms a bounded path $\in \mathrm{int}K_{\rm H}\times\mathrm{int}K_{\rm H}\times\Re^m$. Any accumulation point of the path $P$ is an asymptotically complementary solution of (HMCP).*

**(ii)** *If the HMCP has an asymptotically complementarity solution $(x_{\rm H}^*, y_{\rm H}^*, z^*) = (x^*,\tau^*,y^*,\kappa^*,z^*)$ with $\tau^*>0$ ($\kappa^*>0$), the any accumulation point $(x_{\rm H}(0), y_{\rm H}(0), z(0)) = (x(0),\tau(0),y(0),\kappa(0),z(0))$ of the bounded path $P$ satisfies $\tau(0)>0$ ($\kappa(0)>0$).*

The above two theorems imply that if we have an accumulation point of the central path $P$, we can find that the original MCP falls into exactly one of the three categories in Table 1.

| $\tau^*/\kappa^*$ | $=0$ | $>0$ |
|---|---|---|
| $=0$ | other cases | strongly infeasible |
| $>0$ | solvable | N/A |

Note that the assumption in the theorems is slightly ambiguous since it is concerned with the homogeneous map $\psi_{\rm H}$, but not with the map $\psi$ appearing in the original MCP. The following proposition gives a class of $\psi$s for which $\psi_{\rm H}$ satisfies Assumption 2.1.

**Proposition 4.3** **(i)** *The map $\psi_{\rm H}$ is monotone on $\mathrm{int}K_{\rm H}\times\Re^m$ whenever $\psi$ is monotone on $K\times\Re^m$.*

**(ii)** *The map $F_{\rm H}$ is $(x_{\rm H},y_{\rm H})$-everywhere-monotone on $\mathrm{int}K_{\rm H}\times\mathrm{int}K_{\rm H}\times\Re^m$ whenever $\psi$ is monotone on $K\times\Re^m$.*

**(iii)** *The map $F_{\rm H}$ is $z$-bounded on $\mathrm{int}K_{\rm H}\times\mathrm{int}K_{\rm H}\times\Re^m$ whenever $\psi$ is affine and $z$-bounded on $K\times\Re^m$.*

**(iv)** *The map $F_{\rm H}$ is $z$-injective on $\mathrm{int}K_{\rm H}\times\mathrm{int}K_{\rm H}\times\Re^m$ whenever $\psi$ is $z$-injective on $K\times\Re^m$.*

# References


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