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A New Approach for Computing Bond Prices by the Hull-White Model with Stepwise Reversion Function

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1 Convergence of the Ehrenfest Process to the O-U Process

As shown in Sumita, Gotoh and Jin [3], a sequence of Ehrenfest processes $\{ N_{2V}(t) : t \geq 0 \}$ with approriate scaling and shifting given by

$$X_V(t) \stackrel{\text{def}}{=} \sqrt{\frac{2}{V}} N_{2V}(t) - \sqrt{2V}, \quad V = 1, 2, 3, \cdots \quad (1.1)$$

converges in law to the O-U process $\{X_{\text{OU}}(t) : t \geq 0\}$ as $V \to \infty$. Here $\{X_{\text{OU}}(t) : t \geq 0\}$ is governed by

$$\frac{\partial}{\partial t} f(x,t) = \frac{\partial^2}{\partial x^2} f(x,t) + x \frac{\partial}{\partial x} f(x,t) + f(x,t) \quad . \quad (1.2)$$

Numerical algorithms for computing the corresponding transition probabilities $p_{2V:mn}(t)$, $m, n \in \mathcal{N} = \{0, 1, 2, \cdots, 2V\}$ have been also developed in the same paper based on the spectral structure of the Ehrenfest process. As an alternative procedure to this approach, the uniformization procedure of Keilson [2] is employed in this paper.

2 O-U Process in Hull-White Model

In the area of financial engineering, the movement of spot rate is often represented by the O-U process. One example is the Hull-White model (extended Vasicek model). The Hull-White model is a one factor term structure model characterized by

$$d\hat{x} = (\phi(t) - \alpha(t)\hat{x})dt + \sigma(t)dW_t \quad , \quad (2.1)$$

where $\hat{x}$ is the short rate, W_t is a Wiener process, $\phi(t)$ is the drift, $\sigma(t)$ is the volatility, and $\alpha(t)$ is the reversion function.
For a special case with $\phi(t) = 0$, Equaiton (2.1) becomes

$$d\tilde{x} = -\alpha(t)\tilde{x}dt + \sigma(t)dW_t \quad , \quad (2.2)$$

and the original O-U process $\{ X_{\text{OU}}(t) : t \geq 0 \}$ of (1.2) is a particular case of (2.2) with

$$dx = -xdt + \sqrt{2}dW_t \quad (2.3)$$

where $\alpha(t) = 1$ and $\sigma(t) = \sqrt{2}$.
Let $\hat{X}_{OU}(t)$ be the O-U process in (2.1) and $X_{OU}(t)$ be the O-U process in (2.3), when $\phi(t)$, $\alpha(t)$ and $\sigma(t)$ are constant, the relation between $\hat{X}_{OU}(t)$ and $X_{OU}(t)$ is

$$\hat{X}_{OU}(t) = \rho(t)X_{OU}(t) + \xi(t) \quad , \quad (2.4)$$

where

$$\begin{cases} \rho(t) &= \sigma\sqrt{\frac{1-e^{-2\alpha t}}{2\alpha(1-e^{-2t})}} \\ \xi(t) &= \theta(t) - \rho(t)x(0)e^{-t} + \tilde{x}(0)e^{-\alpha t} \\ \theta(t) &= \frac{\phi}{\alpha}(1-e^{-\alpha t}) + \theta(0)e^{-\alpha t} \end{cases} .$$

Suppose $x(0) = \tilde{x}(0) = 0$, then

$$\hat{X}_{OU}(t) = \rho(t)(t)X_{OU}(t) + \theta(t) \quad . \quad (2.5)$$

The transition probability of $\hat{X}_{OU}(t)$ is the same as that of $X_{OU}(t)$ though they have different state spaces. The relation of (2.4) is obtained from the conditional density function of $\hat{X}_{OU}(t)$ and $X_{OU}(t)$. The solution of the O-U process in (2.1) is well known and is given by

$$\hat{x}(t) = \hat{x}(0)e^{-\int_0^t \alpha(s)ds} + \int_0^t \phi(\tau)e^{-\int_\tau^t \alpha(s)ds}d\tau + \int_0^t \sigma(\tau)e^{-\int_\tau^t \alpha(s)ds}dW_\tau \quad . \quad (2.6)$$

Let $\hat{x}(t) = \tilde{x}(t) + \theta(t)$ with $\tilde{x}(0) = 0$, then the above equation becomes

$$\hat{x}(t) = \tilde{x}(0)e^{-\int_0^t \alpha(s)ds} + \int_0^t \sigma(\tau)e^{-\int_\tau^t \alpha(s)ds}dW_\tau + \theta(t) \quad (2.7)$$

where

$$\theta(t) = \hat{x}(0)e^{-\int_0^t \alpha(s)ds} + \int_0^t \phi(\tau)e^{-\int_\tau^t \alpha(s)ds}d\tau. \quad (2.8)$$

Accordingly the corresponding conditional transition denstiy function of (2.1), (2.2) and (2.3) can be obtained. Consistently, when both $\alpha(t)$ and $\sigma(t)$ are constant, one can get Equation (2.4) immediately.

3 Discount Bond Pricing

The Hull-White model is considered to be one of the most reasonable models for practitioners to evaluate interest rate option. In this study we focus on the price of discount bond. According to Hull and White [1], there exists an explicit formula for pricing a pure discount bond. But this formula is not applied when $\alpha(t)$ is a time-dependent function. In this case, $r(t)$ is needed but difficult to know. The behavior of $r(t)$ can only be grasped through the numerical approximation. Let $D(k,m|K,\alpha)$ be the discount bond price at time $k\tau$ and state m with maturity $K\tau$, where τ is the length of each time step and α is the constant reversion function. Then it can be evaluated by

$$D(k,m|K,\alpha) = e^{-r(k,m)\tau} \sum_{i\in\mathcal{N}} p_{2V:mi}(\tau) D(k+1,i|K) \ , \tag{3.1}$$

where $dr = (\phi - \alpha r)dt + \sigma dW_t$.
Suppose $dr = (\phi - \alpha(t)r)dt + \sigma dW_t$, where $\alpha(t)$ is a step function:

$$\alpha(t) = \begin{cases} \alpha_1 & 0 \le t \le T_1 \\ \alpha_2 & T_1 \le t \le T_2 \end{cases} \ , \tag{3.2}$$

then the pure discount bond price at time t ($0 \le t \le T_1$) with maturity T_2 can be obtained by the following approach.

1. Discretize the time axis as

$$\{0, \tau_1, 2\tau_1, \cdots, K_1\tau_1, T_1+\tau_2, \cdots, K_2\tau_2\} \ ,$$

where $\tau_1 = \frac{T_1}{K_1}$ and $\tau_2 = \frac{T_2-T_1}{K_2-K_1}$.

2. Evaluate the discount bond price at time T_1:

$$D(K_1, n|K_2, \alpha_2)$$

3. Evaluate the discount bond price at time $t = k\tau_1$:

$$D(k, m|K_2, \alpha_1)$$

Accordingly, numerical algorithms can be easily developed. Figure 3.1 depicts the behaviors of convergence with respect to time step K, where the discount bond prices are evaluated with reversion function

$$\alpha(t) = \begin{cases} 0.2 & 0 \le t \le 2 \\ 0.1 & 2 \le t \le 5 \end{cases}$$

Other parameters are fixed as $\phi = 0.05$, $r(0) = 0.05$, and $\sigma = 0.01$. In the figure, "HW trinomial tree" indicates the Hull-White trinomial procedure while "OU procedure" indicates the proposed procedure.

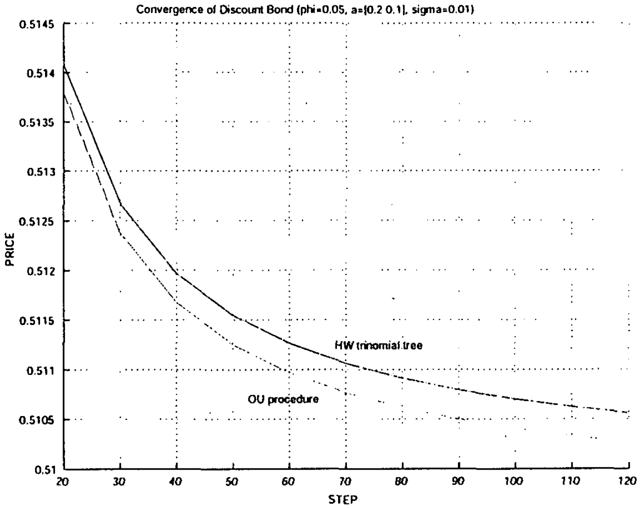

Figure 3.1: Behaviors of convergence of discount bond

4 Conclusion and Futrue Work

Numerical experiments show that the performance of our proposed procedure based on the Ehrenfest process approximation is better than that of the Hull-White trinomial tree for a certain range of parameter values. The present paper only evaluates discount bond price with step function of two steps. Future work will develop the procedure to the step function with multi-steps which is considered to be reasonable in the real situation. Furthermore, the Hull-White model is aptitudinal for interest rate option pricing as well known, so we will extend this procedure to interest rate option pricing.

References

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[3] Sumita, U., J. Gotoh and H. Jin, "Numerical Exploration of Dynamic Behavior of the Ornstein-Uhlenbeck Process via Ehrenfest Process Approximation", Discussion paper series, No.1050, Institute of Policy and Planning Sciences, University of Tsukuba, 2003.